DECORATION INVARIANTS FOR HORSESHOE BRAIDS

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Abstract. The Decoration Conjecture describes the structure of the set of braid types of Smale's horseshoe map ordered by forcing, providing information about the order in which periodic orbits can appear when a horseshoe is created. A proof of this conjecture is given for the class of so-called *lone* decorations, and it is explained how to calculate associated braid conjugacy invariants which provide additional information about forcing for horseshoe braids.

1. Introduction

Forcing relations are a valuable tool in the dynamical study of parameterized families of transformations: they provide information about when the presence of certain dynamical features, such as the existence of periodic orbits of a particular type, imply the presence of other dynamical features.

For surface homeomorphisms, Smale's horseshoe is the paradigmatic map with complicated dynamical behaviour, and understanding how it is created in parameterized families of homeomorphisms is an important problem. In this context, it is fruitful to study the forcing order on the set of braid types of horseshoe periodic orbits [Boy84]: this partial order describes constraints on the order in which periodic orbits can appear during the creation of a horseshoe.

Algorithmic implementations [BH95, FM93, Los93] of Thurston's classification of surface homeomorphisms provide a means of deciding whether or not one given braid type forces another, but such an approach doesn't provide any information about the global structure of the forcing order on the set of all horseshoe braid types.

The *decoration conjecture* [dCH02a] claims that the set of horseshoe braid types is partitioned into families $\mathcal{D}^w = \{\beta_q^w\}$, each parameterized by a rational number q, which are totally ordered by the forcing order, in such a way that β_q^w forces $\beta_{q'}^w$ if and only if $q \leq q'$. The families are labelled by *decorations* w, which are finite words in the symbols 0 and 1. Within families this trivializes the computation of forcing: simply compare the rational parameters in the usual order (or, equivalently, compare the symbolic representations of the braids using the unimodal order).

Some special cases of this conjecture have been proved, and it is supported by strong intuitive evidence from pruning theory: however, a general proof has so far been elusive. In this paper, the conjecture is proved for a class of decorations called *lone*. There are many lone decorations: two infinite families of them are described

2000 *Mathematics Subject Classification.* 37E30, 37E15, 37B10, 37B40.
Key words and phrases. Horseshoe, Forcing relations, Decoration conjecture.

in Sections 6.2 and 6.3 below. Collins [Col05] states that 21 of the 63 decorations of length 5 or less are lone[1].

The proofs of the main results presented here combine the pruning techniques introduced in [dC99] with unremovability arguments of the type developed in [Hal94]. Although pruning has provided an inspiration for several previous results about forcing, this seems to be the first time that it has been used effectively in proving such results.

The second goal of the paper is to present a new set of braid type invariants. Each totally ordered family $\mathcal{D}^w = \{\beta_q^w\}$ gives rise to an invariant (or, looking at it from the point of view of the horseshoe braids themselves, a braid conjugacy invariant) r^w, defined on the set of all horseshoe braid types β by

$$r^w(\beta) = \inf\{q \,:\, \beta_q^w \leq \beta\}.$$

These invariants provide much additional information about the forcing order: a periodic orbit of braid type β can only be created once periodic orbits of braid types β_q^w have been created for all $q > r^w(\beta)$. Notice in this statement that, while β_q^w is restricted to be a horseshoe braid of lone decoration, β can be *any* horseshoe braid.

The techniques presented here make it possible, for lone decorations, both to prove that the family $\mathcal{D}^w$ is totally ordered by forcing, and to calculate the associated *decoration invariant* r^w.

The necessary background material on horseshoe braids and the decoration conjecture is given in Section 2, before the main theorem and the algorithm for computing decoration invariants are presented in Section 3. Some further background material required for the proof, mostly concerning pruning and the Asimov-Franks theorem, is given in Section 4. The proof of the main theorem is given in Section 5. This is followed by some examples and applications in Section 6. The applications include:

- a treatment of the so-called "Star" decorations (Section 6.2), which were introduced in [dCH04], including the completion of the main theorem of that paper (Theorem 54 here), thus providing a description of forcing *between* the families corresponding to different star decorations;
- an example of how decoration invariants can be used to prove that certain other decorations are lone, and hence provide their own invariants (Theorem 59);
- an example of how decoration invariants can be used to prove that certain horseshoe braids are of pseudo-Anosov type (Theorem 62); and
- a discussion of topological entropy bounds arising from decoration invariants (Section 6.4).

[1]These are: the empty decoration $\cdot$, 0, 1, 00, 11, 000, 111, 101, 0000, 0110, 1111, 1001, 00000, 01001, 11001, 10010, 10011, 11011, 11111, 10101, and 10001.

2. Horseshoe braids: height and decoration

This section contains the background material necessary to understand the statement of the main theorem (Theorem 19). Although complete, the treatment is rather terse: the papers [dCH02a, dCH02b] are recommended for readers seeking a more detailed account.

2.1. Smale's horseshoe and the unimodal order. In this paper, the standard model of Smale's horseshoe map [Sma67] $F : D^2 \to D^2$ depicted in Figure 1 is used. The set

$$\Lambda = \{x \in D^2 \,:\, F^n(x) \in S \text{ for all } n \in \mathbb{Z}\}$$

(where S is the square depicted in Figure 1) is a Cantor set, and the *itinerary map* $k : \Lambda \to \{0,1\}^{\mathbb{Z}}$ defined by

$$k(x)_i = \begin{cases} 0 & \text{if } F^i(x) \in H_0 \\ 1 & \text{if } F^i(x) \in H_1 \end{cases}$$

is a homeomorphism, conjugating $F|_\Lambda : \Lambda \to \Lambda$ to the *shift map* $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$.

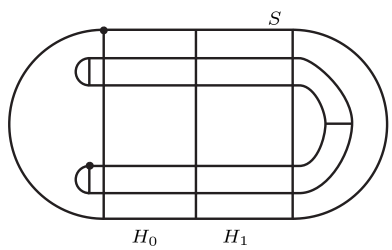

Figure 1. Smale's horseshoe map

The *unimodal order* $\preceq$ is a total order defined on $\{0,1\}^{\mathbb{N}}$ as follows: if $s, t \in \{0,1\}^{\mathbb{N}}$, then $s \preceq t$ if and only if either $s = t$, or the word $s_0 s_1 \dots s_i$ contains an even number of 1s, where i is least such that $s_i \neq t_i$.

This order reflects the horizontal and vertical ordering of points $x, y \in \Lambda$. Define the *horizontal and vertical coordinate functions* $h, v : \Lambda \to \{0,1\}^{\mathbb{N}}$ by $h(x) = k(x)_0 k(x)_1 k(x)_2 \dots$ and $v(x) = k(x)_{-1} k(x)_{-2} k(x)_{-3} \dots$. Then x lies to the left of y if and only if $h(x) \prec h(y)$, and x lies below y if and only if $v(x) \prec v(y)$.

2.2. Notation. Much of the technical part of the paper is concerned with constructing elements of $\{0,1\}^{\mathbb{N}}$ or $\{0,1\}^{\mathbb{Z}}$ from words $w \in \{0,1\}^j$ for various j. The following notation will be used.

A *word* is an element of $\bigcup_{j \geq 0} \{0,1\}^j$.

Let $u = u_0 u_1 \dots u_{j-1} \in \{0,1\}^j$ be a word. Then $|u| = j$ denotes the length of u. u is said to be *even (odd)* if it contains an even (odd) number of 1s. Let $\hat{u} = u_{j-1} \dots u_1 u_0$ denote the reverse of u, $\breve{u} = (1-u_0) u_1 \dots u_{j-1}$ denote u with the initial symbol changed, and $\tilde{u} = u_0 u_1 \dots (1-u_{j-1})$ denote u with the final symbol changed. When two of these accents are combined, they are applied 'bottom upwards': thus,

for example, $\widehat{\widehat{u}} = \breve{u}$ is the word obtained by changing the final symbol of u and then reversing the result. Denote by u^+ the word $u_0u_1 \dots u_{j-1}u_j \in \{0,1\}^{j+1}$, where u_j is chosen so that u^+ is even; and by ${}^+u$ the word $u_{-1}u_0u_1 \dots u_{j-1} \in \{0,1\}^{j+1}$, where u_{-1} is chosen so that ${}^+u$ is even.

$\overline{u}$ denotes the element $\dots uuu \cdot uuu \dots$ of $\{0,1\}^{\mathbb{Z}}$ and u^∞ denotes the element $uuu\dots$ of $\{0,1\}^{\mathbb{N}}$. If v, w, x are also words, then ${}^\infty uv \cdot wx^\infty$ denotes the element $\dots uuuv \cdot wxxx \dots$ of $\{0,1\}^{\mathbb{Z}}$. If $b = b_0b_1b_2\dots$ and $f = f_0f_1f_2\dots$ are elements of $\{0,1\}^{\mathbb{N}}$, then $b \cdot f$ denotes the element $\dots b_2b_1b_0 \cdot f_0f_1f_2\dots$ of $\{0,1\}^{\mathbb{Z}}$. Similarly $bu{\cdot}vf$ denotes the element $\dots b_2b_1b_0u \cdot vf_0f_1f_2\dots$ of $\{0,1\}^{\mathbb{Z}}$, uf denotes the element $uf_0f_1f_2\dots$ of $\{0,1\}^{\mathbb{N}}$, and so on.

A *non-empty initial subword* of $u \in \{0,1\}^j$ is a word $u_0 \dots u_{i-1} \in \{0,1\}^i$ for some i with $0 < i \le j$. Similarly, a *non-empty final subword* of u is a word $u_iu_{i+1}\dots u_{j-1}$ for some i with $0 \le i < j$.

2.3. **The horseshoe and its inverse.** Recall (see e.g. [dCH03]) that F is conjugate to its inverse: $F^{-1} = \phi \circ F \circ \phi^{-1}$, where $\phi : D^2 \to D^2$ is the (orientation-reversing) homeomorphism obtained by first reflecting S in its horizonal centre line, and then rotating it anticlockwise about its centre point through an angle $\pi/2$.

The involution ϕ restricts to an involution $\Lambda \to \Lambda$ which corresponds to reversing itineraries: if $k(x) = b \cdot f$ then $k(\phi(x)) = f \cdot b$.

2.4. **Height.** The *height function* [Hal94] is a (not strictly) decreasing function $\{0,1\}^{\mathbb{N}} \to [0,1/2]$ which is central to the results and methods of this paper. In order to define it, is is necessary first to introduce, for each rational $m/n \in (0,1/2]$, a word $c_{m/n} \in \{0,1\}^{n+1}$: these words will also play a central rôle throughout the paper.

Definition 1. Let m/n be a rational in $(0,1/2]$. Let $L_{m/n}$ be the straight line in $\mathbb{R}^2$ from $(0,0)$ to (n,m). For $0 \le i \le n$, let $s_i = 1$ if $L_{m/n}$ crosses some line $y =$ integer for $x \in (i-1,i+1)$, and $s_i = 0$ otherwise. Then the word $c_{m/n} \in \{0,1\}^{n+1}$ is defined by $c_{m/n} = s_0s_1\dots s_n$.

So, for example, $c_{3/10} = 10011011001$ can be read off from Figure 2. The general form of these words can be seen in Table 1, which shows $c_{m/n}$ for all $m/n \in (0,1/2)$ with $1 \le m \le 4$ and $3 \le n \le 11$. Note that the words c_q are clearly palindromic, and $c_{m/n}$ is of the form $10^{\kappa_1}1^20^{\kappa_2}1^2\dots1^20^{\kappa_m}1$ for some integers $\kappa_i \ge 0$ (an explicit formula for κ_i can be found in [Hal94], but is not needed here: note, however, that each κ_i is equal either to κ_1 or to $\kappa_1 - 1$). It can also be seen easily from this description (see Lemma 2.7 of [Hal94]) that if $m/n < m'/n'$, then $(c_{m'/n'}0)^\infty \prec (c_{m/n}0)^\infty$.

Definition 2. Let $c \in \{0,1\}^{\mathbb{N}}$. The *height* $q(c)$ of c is the unique element of $[0,1/2]$ with the property that $c \prec (c_q0)^\infty$ for all rationals $q \in (0,q(c))$, and $(c_q0)^\infty \prec c$ for all rationals $q \in (q(c),1/2]$.

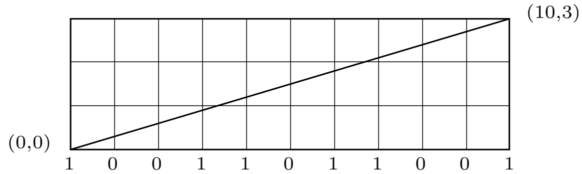

FIGURE 2. $c_{3/10} = 10011011001$

	1	2	3	4
3	1001			
4	10001			
5	100001	101101		
6	1000001			
7	10000001	10011001	10111101	
8	100000001		101101101	
9	1000000001	1000110001		1011111101
10	10000000001		10011011001	
11	100000000001	100001100001	100110011001	101101101101

TABLE 1. Examples of the words $c_{m/n}$ $(1 \leq m \leq 4,\ 3 \leq n \leq 11)$

It is clear from the definition that $q : \{0,1\}^{\mathbb{N}} \to [0,1/2]$ is decreasing. The next lemma (Theorem 3.2 of [Hal94]) provides a practical means of calculating $q(c)$ for all $c \in \{0,1\}^{\mathbb{N}}$ which contain the subword 010 (this is all that will be required in this paper).

Lemma 3. *Let $c \in \{0,1\}^{\mathbb{N}}$, and suppose that c contains the subword* 010*. Then $q(c)$ is rational, and can be calculated as follows. First, if* 10 *is not an initial subword of c, then $q(c) = 1/2$. If* 10 *is an initial subword, then write*

$$c = 10^{\kappa_1}1^{\mu_1}0^{\kappa_2}1^{\mu_2}\dots,$$

where each $\kappa_i \geq 0$, each μ_i is either 1 or 2, and $\mu_i = 1$ only if $\kappa_{i+1} > 0$ (thus κ_i and μ_i are uniquely determined by c). For each $r \geq 1$, define

$$I_r(c) = \left(\frac{r}{2r + \sum_{i=1}^{r} \kappa_i}, \frac{r}{2r - 1 + \sum_{i=1}^{r} \kappa_i} \right],$$

and let $s \geq 1$ be the least integer such that either $\mu_s = 1$, or $\bigcap_{r=1}^{s+1} I_r(c) = \emptyset$. Write $(x, y] = \bigcap_{r=1}^{s} I_r(c) \neq \emptyset$. Then

$$q(c) = \begin{cases} x & \text{if } \mu_s = 2 \text{ and } w \leq x \text{ for all } w \in I_{s+1}(c), \\ y & \text{if } \mu_s = 1, \text{ or } \mu_s = 2 \text{ and } w > y \text{ for all } w \in I_{s+1}(c). \end{cases}$$

Example 4. Notice that the fact that c contains the subword 010 means that $\mu_s = 1$ for some s, and hence the algorithm terminates. It may also terminate before reaching the subword 010. For example, let $c = 1011110011\dots$. Then $\kappa_1 = 1$, $\mu_1 = 2$, $\kappa_2 = 0$, $\mu_2 = 2$, $\kappa_3 = 2$, and $\mu_3 = 2$. This gives $I_1(c) = (1/3, 1/2]$, $I_2(c) = (2/5, 2/4]$, and $I_3(c) = (3/9, 3/8]$. Since $3/8 < 2/5$, the algorithm terminates and $q(c) = 2/5$.

The following technical lemma (which follows immediately from Lemma 63 of [dCH04]) will be needed:

Lemma 5. *Let m/n be a rational in $(0, 1/2]$, and let κ_i be integers such that*

$$c_{m/n} = 10^{\kappa_1}1^2 0^{\kappa_2}1^2 \dots 1^2 0^{\kappa_m}1.$$

Let $1 \le r \le m$, and let f be any element of $\{0,1\}^{\mathbb{N}}$. Then

$$q\left(10^{\kappa_r+1}1^2 0^{\kappa_{r+1}}1^2 \dots 1^2 0^{\kappa_m}1f\right) \le \frac{m}{n}.$$

2.5. **Periodic orbits of the horseshoe.** Let P be a period n orbit of the horseshoe $F : D^2 \to D^2$ (throughout this paper, "period n" means *least* period n). Let p be the rightmost point of P: thus $k(p) = \overline{c_P}$ for some word c_P of length n, which is called the *code* of the periodic orbit P. Note that the choice of p as the rightmost point of P means that

$$\sigma^i(\overline{c_P}) \prec \overline{c_P} \qquad \text{for } 1 \le i < n.$$

Recall [Boy84] that the *braid type* $\mathrm{bt}(P\,;\,f)$ of a period n orbit P of an orientation-preserving homeomorphism $f : D^2 \to D^2$ is a conjugacy class in the mapping class group $\mathrm{MCG}(D_n)$ of the n-punctured disk D_n, namely the conjugacy class of the isotopy class of $h^{-1}fh : D_n \to D_n$, where $h : D_n \to D^2 \setminus P$ is any orientation-preserving homeomorphism (if $P \subseteq \partial D^2$, then first extend f over an exterior collar). Braid types can thus be classified using the Thurston classification [Thu88] as *finite order*, *reducible*, or *pseudo-Anosov.* The *forcing order* $\le$ on the set BT of all braid types is a partial order defined as follows: if $\beta, \gamma \in \mathrm{BT}$, then $\beta \le \gamma$ if and only if every orientation preserving homeomorphism $f : D^2 \to D^2$ which has a periodic orbit P with $\mathrm{bt}(P\,;\,f) = \gamma$ also has a periodic orbit Q with $\mathrm{bt}(Q\,;\,f) = \beta$.

If P is a periodic orbit of the horseshoe, then the symbol P will often be used to denote the braid type $\mathrm{bt}(P\,;\,F)$ as well as the periodic orbit itself. In particular, the notation $P \le Q$ is used as a shorthand for $\mathrm{bt}(P\,;\,F) \le \mathrm{bt}(Q\,;\,F)$.

Most periodic orbits P of the horseshoe are *paired*, in the sense that $\widetilde{c}_P$ is also the code of a horseshoe periodic orbit $\widetilde{P}$: in this case $\mathrm{bt}(P\,;\,F) = \mathrm{bt}(\widetilde{P}\,;\,F)$. Thus, for example, the two periodic orbits P and $\widetilde{P}$ with codes $c_P = 10010$ and $c_{\widetilde{P}} = 10011$ have the same braid type, and it is common to write $c_P = 1001^0_1$, reflecting the fact that the object of interest is the braid type rather than the periodic orbit itself. The only orbits which are not paired are those of even period $2k$, whose codes are of the form $c_P = w\tilde{w}$ for some word w of length k.

The *height* $q(P) \in (0, 1/2]$ of a horseshoe periodic orbit P is defined to be $q(c_P^\infty)$. It is a braid type invariant [Hal94], so in particular $q(P) = q(\widetilde{P})$ if P is paired, and (taking the code which ends with 0), $q(P)$ is rational and can be calculated using the algorithm of Lemma 3. If P is not paired, then again c_P^∞ contains the word 010, so the algorithm terminates and $q(P)$ is rational.

Periodic orbits of the horseshoe can be classified as follows:

Orbits of finite order braid type: There are two fixed points, with codes 0 and 1 (and a fixed point outside of S). There is one period two orbit, with code 10.

For each rational $m/n \in (0, 1/2)$, there is exactly one pair of period n orbits of finite order braid type whose rotation number about the fixed point of code 1 is m/n. The codes of these orbits are $d_{m/n}{}^0_1$, where $d_{m/n}$ is the word consisting of the first $n-1$ symbols of $c_{m/n}$. These periodic orbits have height m/n. There are no other periodic orbits of finite order braid type. (These results are due to Holmes and Williams [HW85].)

NBT orbits: For each rational $q = m/n \in (0, 1/2)$, the words $c_{m/n}{}^0_1$ are the codes of a pair of period $n+2$ orbits of pseudo-Anosov braid type, called *NBT orbits* and denoted P_q^* [Hal94]. These periodic orbits have height q. There are no other horseshoe periodic orbits of these braid types. For $q = 1/2$, the word $c_{1/2}1 = 1011$ is the code of a periodic orbit of reducible braid type, denoted $P_{1/2}^*$.

Orbits described by height and decoration: All other horseshoe periodic orbits P can be described by their height $q(P) \in (0, 1/2] \cap \mathbb{Q}$ and their *decoration*, which is a word $w \in \{0, 1\}^k$ for some $k \geq 0$: a periodic orbit of height q has decoration w if and only if it has one of the four codes $c_q{}^0_1 w^0_1$. Two periodic orbits of the same height and decoration have the same braid type [dCH03]: the notation P_q^w can therefore be used for any periodic orbit of height q and decoration w.

Only certain heights are compatible with a given decoration w. Define the *scope* q_w of w to be the height of the horseshoe periodic orbit containing the point with itinerary $\overline{10w0}$, that is,

$$q_w = \min_{0 \leq i \leq k+2} q\left(\sigma^i((10w0)^\infty)\right).$$

The following result can be found in [dCH02a].

Lemma 6. *For $q < q_w$ there are four periodic orbits of height q and decoration w (i.e. those with codes $c_q{}^0_1 w^0_1$), while for $q > q_w$ there are no periodic orbits of height q and decoration w. (For $q = q_w$, the four words $c_q{}^0_1 w^0_1$ may or may not be codes of periodic orbits of height q.)*

It is convenient for many purposes to consider $*$ to be a decoration as well (with scope $q_* = 1/2$). Then every periodic orbit P of the horseshoe is either of finite order braid type, or can be described uniquely (up to braid type) by its decoration $w \in \{*\} \cup \bigcup_{k \geq 0} \{0, 1\}^k$ and its height $q \leq q_w$. (If $q(P) = m/n$, then the period N of P is either n, or greater than or equal to $n+2$. If $N = n$, then P is of finite order braid type; if $N = n+2$, then P has decoration $*$; and if $N \geq n+3$, then P has decoration w of length $N - (n+3)$.)

The following lemma, which will be used several times, follows immediately from the definition of q_w above, and the fact that $q(\widehat{P}) = q(P)$ for any horseshoe

periodic orbit P, where $\widehat{P}$ is the periodic orbit containing the point of itinerary $\overline{\widehat{c_P}}$ (Lemma 3.8 of [Hal94]).

Lemma 7. *Let $w \in \bigcup_{k\geq 0}\{0,1\}^k$ be a decoration. Then $q_w = q_{\widehat{w}}$.*

The results presented in this paper are related to the *Decoration Conjecture* [dCH02a]. (Statement c) below is not relevant in this paper, and the definition of *topological train track type* is therefore not given.)

Conjecture 8 (Decoration Conjecture)**.**

a) There is an equivalence relation $\sim$ on the set of decorations, with the property that $\mathrm{bt}(P_q^w\,;\,F) = \mathrm{bt}(P_{q'}^{w'}\,;\,F)$ if and only if $q = q'$ and $w \sim w'$.

b) If $q \neq q_w$ then P_q^w has pseudo-Anosov braid type.

c) For each decoration w, the periodic orbits P_q^w with $0 < q < q_w$ all have the same topological train track type.

d) For each decoration w, the set of braid types of periodic orbits of decoration w is totally ordered by forcing, with $P_q^w \leq P_{q'}^w$ if and only if $q \geq q'$.

e) There is a partial order $\preceq$ on the set of equivalence classes of decorations, with the property that if $q > q'$ and $w \preceq w'$ then $P_q^w \leq P_{q'}^{w'}$.

The main theorem of this paper, Theorem 19 below, concerns d) and e) of Conjecture 8, for a certain class of decorations:

Definition 9. A decoration $w \in \bigcup_{k\geq 0}\{0,1\}^k$ is said to be *lone* if for all $q \in (0, q_w) \cap \mathbb{Q}$, the four horseshoe periodic orbits of height q and decoration w are the only horseshoe periodic orbits of their braid type.

Theorem 19 states that Conjecture 8 d) holds for lone decorations w, when restricted to those P_q^w which have pseudo-Anosov braid type (so if b) is true, then so is d) for lone decorations). It also elucidates the partial order $\preceq$ of e) by providing a means of determining, for any lone decoration w and any horseshoe periodic orbit R, which of the orbits P_q^w are forced by R.

Remark 10. If Conjecture 8 a) holds, then if there is a single $q \in (0, q_w) \cap \mathbb{Q}$ for which the four orbits of height q and decoration w are the only horseshoe periodic orbits of their braid type, then the same is true for all $q \in (0, q_w) \cap \mathbb{Q}$. In this case, the lone decorations are precisely those which are alone in their $\sim$-equivalence classes.

The results stated in the next lemma can all be found in [Hal94]:

Lemma 11. *Let P and Q be periodic orbits of the horseshoe.*

a) Height is a braid type invariant.

b) If $q < q(P)$ then $P_q^* \geq P$, while if $q > q(P)$ then $P_q^* \not\geq P$.

c) If $P \geq Q$ then $q(P) \leq q(Q)$.

3. Statement of results

Let $w \in \bigcup_{k\geq 0}\{0,1\}^k$ be a lone decoration. Let

$$\mathcal{D}^w = \{\mathrm{bt}(P_q^w\,;\,F)\,:\,q \in (0,q_w]\cap\mathbb{Q}\},$$

the set of braid types of horseshoe periodic orbits of decoration w. Let $\mathcal{K}^w$ be the subset of $\mathcal{D}^w$ consisting of pseudo-Anosov braid types: this contains $\mathrm{bt}(P_q^w\,;\,F)$ for a dense set of $q \in (0,q_w]\cap\mathbb{Q}$ by Lemma 18 below (and, if Conjecture 8 b) holds, for all q except possibly q_w).

The main results of this paper are:

- that $\mathcal{K}^w$ is totally ordered by forcing, with $P_q^w \leq P_{q'}^w$ if and only if $q \geq q'$; and
- that there exists a practical algorithm to determine, for any horseshoe periodic orbit R, the number $r^w(R) \in (0,q_w]$ with the property that

$$\begin{aligned} R &\geq P_q^w \qquad \text{if} \quad q > r^w(R), \text{ and}\\ R &\not\geq P_q^w \qquad \text{if} \quad q < r^w(R). \end{aligned}$$

  In particular, for each lone decoration w, r^w is a braid type invariant defined on the set of all horseshoe periodic orbits.

The algorithm to compute $r^w(R)$ is complicated to state, although it is easily implemented and is computationally light[2]. It will therefore be described informally and illustrated by examples first; a formal description which is more suitable for use in later proofs will then be given. The purpose of the algorithm is to decide for which values of q points of R are contained in certain disks bounded by segments of stable and unstable manifold through points of the orbits P_q^w: this is what determines which of the P_q^w are forced by R (Theorem 51).

Let R be any horseshoe periodic orbit, with code c_R: if R is paired, then choose the code ending with 0 (so that the algorithm of Lemma 3 can be used to compute the various heights below). Let w be any word. Then $r^w(R)$ is given by

$$r^w(R) = \min(\lambda^w(R), \max(\mu^w(R), \nu^w(R))),$$

where $\lambda^w(R)$, $\mu^w(R)$, and $\nu^w(R)$ are elements of $(0,q_w]\cap\mathbb{Q}$ which will now be described.

$\mu^w(R)$. Recall that ${}^{+}w$ is the word obtained by prepending one symbol to the front of w, in such a way that ${}^{+}w$ is even. Let v be a non-empty even final subword of ${}^{+}w$, and seek all occurrences of the words $\check{v}{}^{0}_{1}10$ in one period of $\overline{c_R}$ (that is, all occurrences of either $\check{v}010$ or of $\check{v}110$: recall that $\check{v}$ is the word obtained from v by changing its initial symbol). For each such occurrence, compute the height of the forward sequence in $\overline{c_R}$ starting at the final symbols 10 in the occurrence. $\mu^w(R)$ is the minimum of such heights taken over all such occurrences and all non-empty

[2]The *horseshoe calculator* script at `http://www.maths.liv.ac.uk/cgi-bin/tobyhall/horseshoe` implements this algorithm.

even final subwords v of ${}^+w$. If there are no such occurrences, or if the minimum of the heights is greater than q_w, then $\mu^w(R) = q_w$.

Example: Let R have code $c_R = 100010111001010$, and let $w = 1$. Then ${}^+w = 11$, which has only one non-empty even final subword, namely $v = 11$. Hence $\check{v} = 01$, and occurrences of 01010 and 01110 are sought. The three occurrences of such words in one period of $\overline{c_R}$ are shown in Figure 3. The corresponding forwards sequences have heights $q(10010\ldots) = 1/3$, $q(1010\ldots) = 1/2$, and $q(100010\ldots) = 1/4$. Hence $\mu^w(R) = 1/4$.

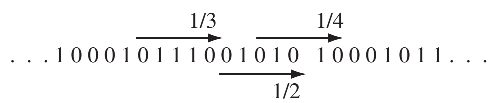

FIGURE 3. Computing $\mu^w(R)$

$\nu^w(R)$**.** The computation of $\nu^w(R)$ is similar: let v be a non-empty even initial subword of w^+, and seek all occurrences of the words $01^0_1\tilde{v}$ in one period of $\overline{c_R}$. For each such occurrence, compute the height of the backward sequence in $\overline{c_R}$ starting at the initial symbols 01 in the occurrence. $\nu^w(R)$ is the minimum of such heights taken over all such occurrences and all non-empty even initial subwords v of w^+. If there are no such occurrences, or if the minimum of the heights is greater than q_w, then $\nu^w(R) = q_w$.

Example: Continuing with the above example, $w^+ = 11$ which has only one non-empty even initial subword, namely $v = 11$. Hence $\tilde{v} = 10$, and occurrences of 01010 and 01110 are sought. This gives $\nu^w(R) = 1/3$ (see Figure 4).

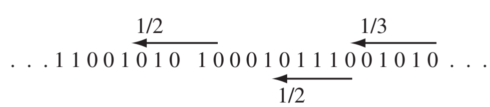

FIGURE 4. Computing $\nu^w(R)$

$\lambda^w(R)$**.** Seek all occurrences of the four words $01^0_1 w^0_1 10$ in one period of $\overline{c_R}$. For each such occurrence, compute $\max(q(b), q(f))$, where b is the backward sequence starting at the initial symbols 01 in the occurrence, and f is the forward sequence starting at the final symbols 10. Then $\lambda^w(R)$ is the minimum value of this quantity taken over all such occurrences. If there are no such occurrences, or if the minimum is greater than q_w, then $\lambda^w(R) = q_w$.

Example: Continuing with the above example, occurrences of 0101010, 0111010, 0101110, and 0111110 are sought. There are two such occurrences, one with $q(b) = 1/3$ and $q(f) = 1/4$, and the other with $q(b) = 1/4$ and $q(f) = 1/3$ (see Figure 5). Hence $\max(q(b), q(f)) = 1/3$ for both occurrences, and hence $\lambda^w(R) = 1/3$.

Thus $r^w(R) = \min(\lambda^w(R), \max(\mu^w(R), \nu^w(R))) = \min(1/3, \max(1/4, 1/3)) = 1/3$.

$$\begin{array}{c} \overset{1/3}{\longleftarrow}\!\!\overset{1/4}{\longrightarrow}\;\overset{1/4}{\longleftarrow}\!\!\overset{1/3}{\longrightarrow} \\ \ldots 0111001010\ 1000101110001010 \ldots \end{array}$$

FIGURE 5. Computing $\lambda^w(R)$

Example 12. Table 2 lists all of the horseshoe periodic orbits of period 8, together with the values of their decoration invariants for the lone decorations of length 3 or less (and for the decoration $*$: see Remark 20). Period 8 orbits with the same height and decoration (and hence the same braid type) are grouped in the same row. Those orbits for which there is not a choice of 4 different codes are either of finite order type (1011011^0_1 and 1000000^0_1), NBT orbits (1000001^0_1), or orbits whose height is equal to the scope of their decoration (10111010, with height 1/2 and decoration 101; 1011111^0_1 with height 1/2 and decoration 111; 1001101^0_1, with height 1/3 and decoration 01; and $10001^0_1 0\binom{0}{1}$, with height 1/4 and decoration 0 — here there are three orbits with this height and decoration, namely those with codes 10001001, 10001100, and 10001101).

This table gives exact information about which of the (infinitely many) orbits with these decorations are forced by which period 8 orbits.

Thus, for example, 7 of the orbit types have $r^{111} = 1/2 = q_{111}$, and hence force no orbits of decoration 111. The other 4 orbit types all force some of the orbits with decoration 111: for example, the orbits of code $10001^0_1 1^0_1$ force P^{111}_q for all $q > 1/4$, and do not force P^{111}_q for all $q < 1/4$.

Decoration	$*$	$\cdot$	0	1	00	11	000	101	111
Scope	1/2	1/3	1/4	1/2	1/5	2/5	1/6	1/2	1/2
10111010	1/2	1/3	1/4	1/2	1/5	2/5	1/6	1/2	1/2
1011111^0_1	1/2	1/3	1/4	1/2	1/5	2/5	1/6	1/2	1/2
1011011^0_1	1/2	1/3	1/4	1/2	1/5	2/5	1/6	1/2	1/2
$1001^0_1 11^0_1$	1/2	1/3	1/4	1/2	1/5	1/3	1/6	1/2	1/2
$1001^0_1 10^0_1$	1/3	1/3	1/4	1/3	1/5	1/3	1/6	1/3	1/3
1001101^0_1	1/3	1/3	1/4	1/3	1/5	1/3	1/6	1/3	1/3
$10001^0_1 0\binom{0}{1}$	1/2	1/3	1/4	1/2	1/5	2/5	1/6	1/2	1/2
$10001^0_1 1^0_1$	1/2	1/4	1/4	1/4	1/5	1/4	1/6	1/2	1/4
100001^{00}_{11}	1/2	1/5	1/5	1/2	1/5	2/5	1/6	1/2	1/2
1000001^0_1	1/6	1/6	1/6	1/6	1/6	1/6	1/6	1/6	1/6
1000000^0_1	1/2	1/3	1/4	1/2	1/5	2/5	1/6	1/2	1/2

TABLE 2. Examples of some decoration invariants for period 8 orbits

On the whole, the orbits in this table force relatively few of the orbits of the given decorations. This is because the period is low. The following "weak universality" result says that, for a fixed decoration w, most horseshoe periodic orbits R have $r^w(R) < q^w$: indeed, most have $r^w(R)$ very small, and hence force almost all orbits of decoration w.

Theorem 13. *Let w be a decoration and $q \in (0, q_w)$. Let p_n denote the proportion of period n horseshoe orbits R with $r^w(R) < q$. Then $p_n \to 1$ as $n \to \infty$.*

Proof. Pick k with $1/k < q$. Then any R whose code includes the word $0^k 10w010^k$ has
$$r^w(R) \leq \lambda^w(R) < q(10^k \dots) < 1/k < q.$$
□

As a further illustration of the use of these invariants, let $R = P^{10}_{1/6}$ be the periodic orbit of code $c_R = 10000011100$. Computing $r^w(R)$ for the same lone decorations w as in Table 2 gives $r^*(R) = 1/3$, $r^\cdot(R) = 1/3$, $r^0(R) = 1/6$, $r^1(R) = 1/3$, $r^{00}(R) = 1/6$, $r^{11}(R) = 1/3$, $r^{000}(R) = 1/6$, $r^{101}(R) = 1/3$, and $r^{111}(R) = 1/3$. The periodic orbits of these decorations are depicted in Figure 6: for each decoration w, there are periodic orbits P^w_q for each rational q in $(0, q_w)$ (represented by points on the vertical lines), but not for rationals $q > q_w$. The decoration invariants show that all orbits on the thicker parts of the lines are forced by R, but that none of the orbits on the thinner parts of the lines are forced by R (the theorem doesn't state whether or not the orbits represented by points at the transition from thin to thick are forced by R). Notice that since $q(R) = 1/6$, R cannot force any orbit P^w_q with $q < 1/6$ by Lemma 11 c). The decoration 10 of R itself is *not* lone, and is not required to be by Theorem 19.

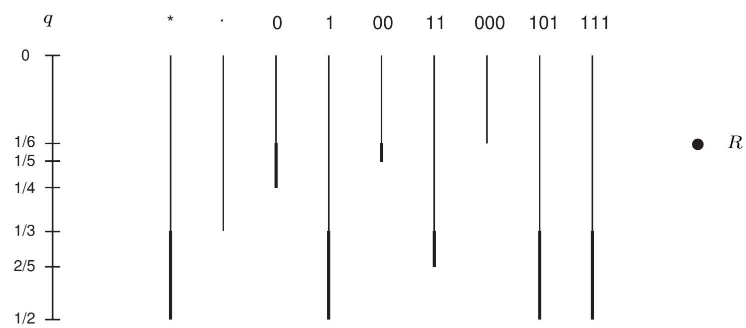

FIGURE 6. Some orbits forced, and not forced, by the orbit of code 10000011100

A formal statement of the algorithm for computing $r^w(R)$ will now be given. Some preliminary definitions are useful.

Definitions 14. Let v and c be words, with $|c| \geq 1$. For each i with $0 \leq i < |c|$, let
$$\overrightarrow{r}(\overline{c}, v, i) = \begin{cases} q(f) & \text{if } \sigma^i(\overline{c}) = bv \cdot f \text{ for some } b, f \in \{0,1\}^{\mathbb{N}} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$
Similarly, define $\overleftarrow{r}(\overline{c}, v, i)$ by replacing $q(f)$ with $q(b)$ in the above, and $\overleftrightarrow{r}(\overline{c}, v, i)$ by replacing $q(f)$ with $\max(q(b), q(f))$.

Then, if V is a finite non-empty set of words, let
$$\overrightarrow{r}(\overline{c}, V) = \min_{v \in V,\ 0 \leq i < |c|} \overrightarrow{r}(\overline{c}, v, i).$$
$\overleftarrow{r}(\overline{c}, V)$ and $\overleftrightarrow{r}(\overline{c}, V)$ are defined similarly using $\overleftarrow{r}(\overline{c}, v, i)$ and $\overleftrightarrow{r}(\overline{c}, v, i)$ in place of $\overrightarrow{r}(\overline{c}, v, i)$.

Finally, if w is a word and R is a horseshoe periodic orbit, define

$$\mu^w(R) = \min\left(q_w, \overrightarrow{r}^w(\overline{c_R}, V)\right),$$

where

$$V = \{\breve{v}_1^0 : v \text{ is a non-empty even final subword of } {}^+w\}.$$

Similarly, define

$$\nu^w(R) = \min\left(q_w, \overleftarrow{r}^w(\overline{c_R}, V)\right),$$

where

$$V = \{{}_1^0\breve{v} : v \text{ is a non-empty even initial subword of } w^+\},$$

and

$$\lambda^w(R) = \min\left(q_w, \overleftrightarrow{r}^w(\overline{c_R}, V)\right),$$

where

$$V = \{0w0, 0w1, 1w0, 1w1\}.$$

Remark 15. It is clear from the definitions that $\lambda^{\hat{w}}(\widehat{R}) = \lambda^w(R)$, $\mu^{\hat{w}}(\widehat{R}) = \nu(R)$, and $\nu^{\hat{w}}(\widehat{R}) = \mu(R)$

The pieces are now all in place to describe the invariant r^w:

Definition 16. Let $w \in \bigcup_{k\geq 0}\{0,1\}^k$ be a decoration, and R be a horseshoe periodic orbit. Then the *w-depth of R*, $r^w(R) \in (0, q_w] \cap \mathbb{Q}$ is defined by

$$r^w(R) = \min\left(\lambda^w(R), \max(\mu^w(R), \nu^w(R))\right).$$

A number of the results presented below require that the braid type of P_q^w be pseudo-Anosov, hence the following definition:

Definition 17. Let w be a decoration. Define

$$\mathbb{Q}^w = \{q \in (0, q_w) \cap \mathbb{Q} : P_q^w \text{ has pseudo-Anosov braid type}\}.$$

Conjecture 8 b) states that $\mathbb{Q}^w = (0, q_w) \cap \mathbb{Q}$ for all decorations w, and this can be proved in a number of special cases. However, in this paper all that is needed is the following straightforward lemma:

Lemma 18. *For any decoration w, $\mathbb{Q}^w$ is dense in $(0, q_w)$.*

Proof. Let w be a decoration of length $k \geq 0$, and let $q = m/n \in (0, q_w)$. Since P_q^w does not have finite order braid type, it must have pseudo-Anosov braid type if its period $n + k + 3$ is prime [Boy84]. Hence

$$\mathbb{Q}^w \supseteq A^w = \{m/n \in (0, q_w) : (m, n) = 1 \text{ and } n + k + 3 \text{ is prime}\}.$$

However, A^w is dense in $(0, q_w)$. Indeed, for any $N \in \mathbb{N}$, the set

$$\mathbb{Q}_N = \{m/n : (m, n) = 1 \text{ and } n + N \text{ is prime}\}$$

is dense in $\mathbb{Q}$. For let $r/s \in \mathbb{Q}$. For each $K \geq 1$, let $r_K = K(N-1)r + 1$ and $s_K = K(N-1)s + 1$, so that $r_K/s_K \to r/s$ as $K \to \infty$, and it suffices to show that each r_K/s_K can be approximated arbitrarily closely by elements of $\mathbb{Q}_N$. Since $r_K s_K$ is coprime to $N - 1$, Dirichlet's theorem gives a strictly increasing sequence

of integers a_K^i such that $a_K^i r_K s_K + (N-1)$ is prime for all i. Let $m_K^i = a_K^i r_K^2$ and $n_K^i = a_K^i r_K s_K - 1$. Then $m_K^i/n_K^i \to r_K/s_K$ as $i \to \infty$, and $m_K^i/n_K^i \in \mathbb{Q}_N$ for all i as required. □

The main theorem of this paper can now be stated:

Theorem 19. *Let w be a lone decoration. Then*

a) The braid types of the orbits P_q^w with $q \in \mathbb{Q}^w$ are totally ordered by forcing, with $P_q^w \geq P_{q'}^w$ if and only if $q \leq q'$.

b) For any horseshoe periodic orbit R,

$$r^w(R) = \sup\{q \in \mathbb{Q}^w \,:\, R \not\geq P_q^w\},$$

and hence r^w is a braid type invariant.

Remark 20. There is a corresponding result for the family $\{P_q^* \,:\, q \in (0,1/2] \cap \mathbb{Q}\}$ of NBT orbits [Hal94]. This family is totally ordered by forcing (with $P_q^* \geq P_{q'}^*$ if and only if $q \leq q'$), and the corresponding braid type invariant r^* can be calculated by

$$r^*(R) = \overleftrightarrow{r}\left(\overline{c_R}, \{0,1\}\right).$$

Remark 21. $r^w(R)$ gives information about which periodic orbits of decoration w are forced by R. Invariants $q^w(R)$ which give information about which periodic orbits of decoration w force R can be defined similarly:

$$q^w(R) = \sup\{q \in \mathbb{Q}^w \,:\, P_q^w \geq R\}$$

(or $q^w(R) = q_w$ if $P_q^w \not\geq R$ for all q). However, the authors know of no means of computing these invariants, except for the NBT decoration $w = *$, for which $q^*(R) = q(R)$ by Lemma 11 b).

4. Linking numbers, the Asimov-Franks theorem, and pruning

The main technique used in this paper to show that the braid type of one periodic orbit P forces the braid type of a second periodic orbit Q is to apply the Asimov-Franks theorem [AF83, Hal91b] to show that Q is *unremovable* in $D^2 \setminus P$. In order to show that the conditions of the Asimov-Franks theorem hold, linking number considerations will be used. The necessary definitions and results are presented in Sections 4.1 – 4.3.

On the other hand, the technique used to show that the braid type of one periodic orbit does *not* force the braid type of a second is to show that all orbits of the second braid type can be *pruned away* by an isotopy supported in the complement of the first orbit. The pruning theorem used to do this [dC99] is stated in Section 4.4.

4.1. **Linking numbers.** Let $f : D^2 \to D^2$ be an orientation-preserving homeomorphism, and $\{f_t\} : \mathrm{id} \simeq f$ be an isotopy (which will be referred to as a *suspension* of f). Recall that, given two distinct periodic orbits P and Q of f, the *linking number* $L(Q,P)$ of Q about P with respect to the suspension $\{f_t\}$ is an integer

which determines the homology class of the suspension $\pi(\bigcup_{s\in[0,1]}(f_s(Q)\times\{s\}))$ of Q in the suspension manifold

$$\left[(D^2\times[0,1])\setminus\bigcup_{s\in[0,1]}(f_s(P)\times\{s\})\right]\Big/(x,0)\sim(x,1).$$

If the suspension $\{f_t\}$ is changed, then the linking number $L(Q,P)$ changes by a multiple of the period n of Q: thus the linking number is well-defined modulo n.

The following two straightforward lemmas can be found in [Hal91a], where they appear as Lemma 1.23 and Corollary 1.25 respectively. The first can be used to show that two periodic orbits have distinct linking numbers about a third periodic orbit P: intuitively, this means that they cannot move together and annihilate under an isotopy relative to P.

Lemma 22. *Let $f\colon D^2\to D^2$ be an orientation preserving homeomorphism which has periodic points q_0 and q_1 of least period n lying on orbits Q_0 and Q_1, and let $\alpha\colon[0,1]\to D^2$ be a path from q_0 to q_1. Let γ be the closed curve $\alpha\cdot(f^n\circ\alpha)^{-1}$. Suppose that P is a periodic orbit of f with no points lying on α, and that γ has winding number $w_\gamma(P)$ about P. Then*

$$L(Q_0,P)=L(Q_1,P)+w_\gamma(P),$$

where the linking numbers are calculated with respect to any fixed suspension of f.

The second lemma can be used to exclude the possibility of a given periodic orbit collapsing onto an orbit of lower period under isotopy.

Lemma 23. *Let $f_i:D^2\to D^2$ be a sequence of orientation-preserving homeomorphisms, converging in the C^0 topology to a homeomorphism $f:D^2\to D^2$. Let $p_i\to p$ and $q_i\to q$ be sequences in D^2 with the property that each p_i lies on a period m orbit P_i of f_i and each q_i lies on a period n orbit Q_i of f_i. If q is a period n/l point of f which does not lie on the f-orbit of p, then $L(Q_i,P_i)$ is a multiple of l for sufficiently large i (with respect to any suspension of f_i).*

4.2. **The Asimov-Franks theorem.** The Asimov-Franks theorem [AF83] gives conditions under which periodic orbits of homeomorphisms $f:M\to M$ persist (or are *unremovable*) under arbitrary isotopy of f. The version given here is from [Hal91b], and is restricted to the case of interest in this paper (unremovability of single periodic orbits of orientation-preserving homeomorphisms of D^2 under isotopy relative to some other periodic orbit).

Definitions 24. Let $f\colon D^2\to D^2$ and $g\colon D^2\to D^2$ be orientation-preserving homeomorphisms having period n points p and q respectively; and let R be a periodic orbit of f in $\mathrm{Int}(D^2)$. Then $(p\,;\,f)$ and $(q\,;\,g)$ are *connected by isotopy rel. R* (denoted $(p\,;\,f)\sim(q\,;\,g)$) if there exists an isotopy $\{f_t\}\colon f\simeq g$ relative to R and a path α in D^2 from p to q, such that $\alpha(t)$ is a period n point of f_t for all t.

$(p\,;\,f)$ is said to be *unremovable in (D^2,R)* if every homeomorphism $g:D^2\to D^2$ which is isotopic to f rel. R has a period n point q with $(p\,;\,f)\sim(q\,;\,g)$.

Notice that being connected by isotopy rel. R is clearly an equivalence relation on the set of all pairs $(p\,;\,f)$, where $f : D^2 \to D^2$ is an orientation-preserving homeomorphism and p is a periodic point of f.

The Asimov-Franks theorem gives three conditions which together ensure the unremovability of $(p\,;\,f)$. The first condition prevents the orbit of p from collapsing onto an orbit of lower period under isotopy.

Definition 25. $(p\,;\,f)$ is said to be *uncollapsible* if, given any sequence of homeomorphisms $g_j : D^2 \to D^2$ which converge in the C^0 topology to $g : D^2 \to D^2$, and a sequence $q_j \to q$ in D^2 such that q_j is a period n point of g_j with $(q_j\,;\,g_j) \sim (p\,;\,f)$ for all j, then q is a period n point of g.

The second condition prevents the orbit of p from falling into the periodic orbit R.

Definition 26. $(p\,;\,f)$ is said to be *separated from* R if $(p\,;\,f) \sim (q\,;\,g)$ implies $q \notin R$.

The final condition is that the total fixed point index of periodic points which could interact with p under isotopy is non-zero. Recall (see for example [Jia83]) that if $f : X \to X$ is a continuous self-map of a compact manifold, then the fixed point index $\mathrm{index}(S, f)$ of a subset S of $\mathrm{Fix}(f)$ can be defined, generalising the familiar notion of the index of an isolated fixed point, provided that S is compact and is open in $\mathrm{Fix}(f)$.

Definition 27. Let p be a period n point of f. The *strong Nielsen class* $\mathrm{snc}(p\,;\,f)$ of $(p\,;\,f)$ is the set of all period n points q of f with $(p\,;\,f) \sim (q\,;\,f)$.

If $(p\,;\,f)$ is uncollapsible and separated from R, then $\mathrm{snc}(p\,;\,f)$ is compact and open in the set of fixed points of f^n, so the following definitions can be made:

Definitions 28. Let $(p\,;\,f)$ be uncollapsible and separated from R. The *index* of $(p\,;\,f)$ is defined by $I(p\,;\,f) = \mathrm{index}(\mathrm{snc}(p\,;\,f), f^n)$. $(p\,;\,f)$ is said to be *essential* if $I(p\,;\,f) \neq 0$.

Theorem 29 (Asimov-Franks)**.** *If $(p\,;\,f)$ is uncollapsible, separated from R, and essential, then it is unremovable.*

The following result can be found in [Hal91b]: it says that relevant topological information is preserved under connection by isotopy.

Lemma 30. *Let $(p\,;\,f) \sim (q\,;\,g)$, and let the orbits of p and q be denoted P and Q respectively. Then*

a) $\mathrm{bt}(P\,;\,f) = \mathrm{bt}(Q\,;\,g)$.

b) If $P \neq R$ and $Q \neq R$, then for any suspension of f there is a suspension of g such that $L(P, R) = L(Q, R)$ with respect to the given suspensions.

Remark 31. By Lemma 30 a), $(p\,;\,f)$ is necessarily separated from R if $\mathrm{bt}(P\,;\,f) \neq \mathrm{bt}(R\,;\,f)$ (and in particular if P and R have different periods).

The following trivial result will also be useful:

Lemma 32. *Let p and q be period n points of f and g. Then $(p\,;\,f) \sim (q\,;\,g)$ if and only if $(f(p)\,;\,f) \sim (g(q)\,;\,g)$.*

4.3. **A method for showing that two periodic points are connected by isotopy.**

Definition 33. Let $f\colon D^2 \to D^2$ be an orientation-preserving homeomorphism with distinct period n points p and q. Suppose that $\alpha\colon [0,1] \to D^2$ is an arc from p to q with the property that

a) there exist $a, b \in [0,1]$ such that
   i) $\alpha([a,1]) \cup f^n(\alpha([b,1]))$ is a simple closed curve bounding a (closed) disk Δ, and
   ii) $\alpha([0,a]) = f^n(\alpha([0,b]))$,
b) $\alpha([0,1])$ is disjoint from $f^i(\alpha([0,1]))$ for $1 \le i < n$, and
c) the orbits P and Q of p and q are disjoint from Int Δ.

Then α is said to *define the disk Δ under f*.

Remarks 34.

a) Note that condition b) implies that $\alpha([0,1])$ intersects $P \cup Q$ only at p and q.
b) Condition a) is ponderous, but the reason for it is simply explained. In the applications in this paper, where f is the horseshoe, the arc α will be taken to be an arc of the stable manifold of p followed by an arc of the unstable manifold of q. Suppose, for example, that p is a periodic point of negative index (so that f^n sends each branch of its stable manifold to itself), while q is a periodic point of positive index (so that f^n sends each branch of its unstable manifold to the other branch). The parameter b corresponds to the point x where the two manifolds meet, so that $f^n(\alpha([0,b]))$ is a subset $\alpha([0,a])$ of the image of α: the remainders $\alpha([a,1])$ and $f^n(\alpha([b,1]))$ of the arc and its image under f^n are disjoint except at their endpoints (which are at x and q), and hence form a simple closed curve. See Figure 8, for example.

Theorem 35. *Let $f\colon D^2 \to D^2$ be an orientation-preserving homeomorphism, p and q be distinct period n points of f, and R be a finite subset of D^2 with $f(R) = R$. Suppose that there exists an arc $\alpha\colon [0,1] \to D^2 \setminus R$ from p to q, which defines a disk Δ under f which is disjoint from R. Then $(p\,;\,f)$ and $(q\,;\,f)$ are connected by isotopy rel. R.*

Proof. Denote by α_i the image of the arc $f^i \circ \alpha$ for $0 \le i \le n$: since f is a homeomorphism, it follows from Definition 33 b) that $\alpha_i \cap \alpha_j = \emptyset$ for $i \ne j$, except if $\{i, j\} = \{0, n\}$.

Thus α_i has endpoints on $P \cup Q$ and is disjoint from $\partial\Delta$ for $1 \le i < n$, hence by Definition 33 c) $\alpha_i \cap \Delta = \emptyset$ for $1 \le i < n$. Let D be a disk which contains $\alpha_0 \cup \alpha_n$ in its interior, but which is disjoint from R and from α_i for $1 \le i < n$; and let C be a simple closed curve bounding a disk containing α_0 in its interior such that both C and $f^n(C)$ are contained in Int D, while $f^i(C)$ is disjoint from D for $1 \le i < n$.

Then $f^n(C)$ is isotopic to C rel. $P \cup Q \cup R$, since both are simple closed curves contained in D, surrounding the only two points of $P \cup Q \cup R$ which lie in D. Hence (by a theorem of Epstein [Eps66]) there is a homeomorphism $h\colon D^2 \to D^2$, supported in D and isotopic to the identity rel. $P \cup Q \cup R$, with $h(f^n(C)) = C$. Let $F = h \circ f$, so that $F^n(C) = C$ and $f \simeq F$ rel. $P \cup Q \cup R$. Clearly $(p\,;\, f) \sim (p\,;\, F)$ and $(q\,;\, f) \sim (q\,;\, F)$ (using the constant paths from p to p and from q to q, and the isotopy $f \simeq F$ rel. $P \cup Q \cup R$), so it remains to show that $(p\,;\, F) \sim (q\,;\, F)$.

Let E_0 be the closed disk bounded by C, write $E_i = F^i(E_0)$ for $1 \le i < n$, and set $E = \bigcup_{i=0}^{n-1} E_i$. Thus E is an F-invariant subset of D^2, disjoint from R, consisting of n mutually disjoint disks. Let $H\colon D^2 \to D^2$ be a homeomorphism supported in E with $H(F^i(q)) = F^i(p)$ for each i, and let $\{H_t\}\colon H \simeq \mathrm{id}$ be an isotopy supported in E. Then $H_t(q)$ is a period n point of $H_t \circ F \circ H_t^{-1}$ for each t, so that $(p\,;\, G) \sim (q\,;\, F)$, where $G = H \circ F \circ H^{-1}$. However G agrees with F on P and outside of E: thus applying the Alexander trick to all of the components of E (which each contain a single point of P), there is an isotopy from G to F relative to $P \cup R$, which provides an isotopy connection $(p\,;\, F) \sim (p\,;\, G)$. Hence $(p\,;\, F) \sim (q\,;\, F)$ as required. $\square$

4.4. **Pruning theory.** Pruning theory provides a means of destroying some of the dynamics of a surface homeomorphism by an isotopy with controlled support. The following definition and theorem are from [dC99], simplified in accordance with the requirements of this paper.

Definitions 36. A *pruning disk* for the horseshoe map $F : D^2 \to D^2$ is a closed topological disk $\Delta \subseteq D^2$ whose boundary is the union of an arc C of stable manifold and an arc E of unstable manifold, intersecting only at their endpoints, which satisfy

$$F^n(C) \cap \mathrm{Int}(\Delta) = F^{-n}(E) \cap \mathrm{Int}(\Delta) = \emptyset \qquad \text{for all } n \ge 1.$$

The common endpoints of the two arcs are called the *vertices* of Δ.

Theorem 37. *Let Δ be a pruning disk for F. Then there exists an isotopy, supported in $\bigcup_{n\in\mathbb{Z}} F^n(\mathrm{Int}(\Delta))$, from F to a homeomorphism F_Δ for which all points of* $\mathrm{Int}(\Delta)$ *are wandering.*

The pruning isotopy therefore destroys all of the dynamics in $\mathrm{Int}(\Delta)$, while leaving untouched any orbits which do not enter $\mathrm{Int}(\Delta)$. In this paper the pruning theorem will be applied in the form of the following corollary:

Corollary 38. *Let w be a lone decoration and $q \in (0, q_w)$. Suppose that there is a pruning disk Δ containing points of all four periodic orbits P_q^w in its interior. If R is a horseshoe periodic orbit disjoint from Δ, then $R \not\geq P_q^w$.*

Proof. The set of braid types of the pruned homeomorphism F_Δ is precisely the set of braid types of periodic orbits of F which are disjoint from Δ. In particular, F_Δ has a periodic orbit of the braid type of R, but none of the braid type of P_q^w. $\square$

5. Proof of the main theorem

Let w be a lone decoration of length k. This decoration will be fixed throughout the section, and therefore the dependence of many objects on w will be suppressed: on the few occasions when it is temporarily important to indicate this dependence, this will be done by means of a superfix w. For the sake of clarity, it is assumed at first that w is even: the modifications necessary in the case of odd w are described in Section 5.5.

5.1. **Iterated arcs.** The proof of the theorem depends on the details of the configuration of the F-images of a collection of arcs joining points of the four periodic orbits of height q and decoration w for each $q \in (0, q_w) \cap \mathbb{Q}$. For the remainder of this subsection, let $q = m/n$ be a fixed element of $(0, q_w)$. For the sake of notational clarity, arcs $\alpha : [0,1] \to D^2$ and their images $\alpha([0,1])$ will not be distinguished carefully; and points $x \in \Omega(F)$ will often be identified with $k(x) \in \{0,1\}^{\mathbb{Z}}$.

Let $p_1 = \overline{c_q 0w0}$, $p_2 = \overline{c_q 0w1}$, $p_3 = \overline{c_q 1w1}$, and $p_4 = \overline{c_q 1w0}$ be the rightmost points on each of the four periodic orbits of height q and decoration w (which have period $n+k+3$). Note that $\operatorname{index}(p_i, F^{n+k+3}) = (-1)^i$, since the words $c_q 0w0$ and $c_q 1w1$ are even and the words $c_q 0w1$ and $c_q 1w0$ are odd.

Let γ and δ be the following arcs connecting these points:

a) γ goes from p_1 to p_2. It is the concatenation of the vertical arc from p_1 to the point ${}^\infty(c_q 0w1) \cdot (c_q 0w0)^\infty$ and the horizontal arc from ${}^\infty(c_q 0w1) \cdot (c_q 0w0)^\infty$ to p_2.
b) δ goes from p_3 to p_4. It is the concatenation of the vertical arc from p_3 to the point ${}^\infty(c_q 1w0) \cdot (c_q 1w1)^\infty$ and the horizontal arc from ${}^\infty(c_q 1w0) \cdot (c_q 1w1)^\infty$ to p_4.

These arcs are depicted schematically in Figure 7. The points p_i are shown with their correct relative horizontal and vertical orderings, calculated using the fact that w is even.

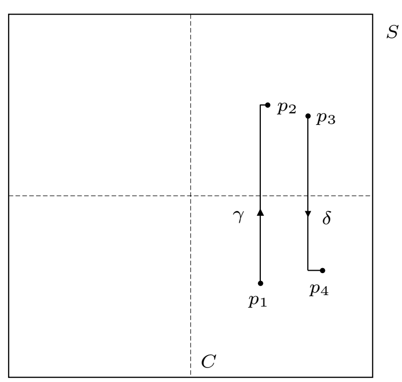

Figure 7. The arcs γ and δ

It will be shown that each of these arcs defines a disk (in the sense of Definition 33) under the horseshoe map F, and explicit conditions for a point of the non-wandering set $\Omega(F)$ to lie in each of the disks thus defined will be given.

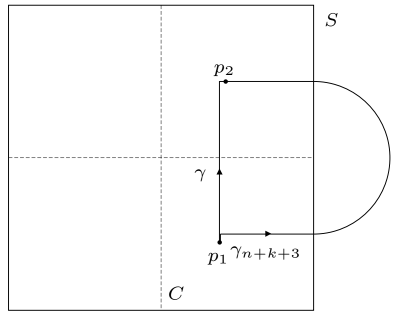

FIGURE 8. The disk defined by γ

Lemma 39. *The arc γ defines a disk C_q under F. A point $x = b \cdot f$ of $\Omega(F)$ lies in* $\operatorname{Int} C_q$ *if and only if*

a) $f \succ (c_q 0 w 0)^\infty$*, and*

b) $\sigma(b) \succ (\hat{w} 0 c_q 1)^\infty$.

Proof. Denote by γ_j the arc $F^j \circ \gamma$, for $0 \le j \le n+k+3$. A straightforward induction shows that for $0 \le j \le n+k+2$, the arc γ_j is contained in S, and is the concatenation of the vertical arc from $\sigma^j(\overline{c_q 0 w 0})$ to $\sigma^j({}^\infty(c_q 0 w 1) \cdot (c_q 0 w 0)^\infty)$ and the horizontal arc from $\sigma^j({}^\infty(c_q 0 w 1) \cdot (c_q 0 w 0)^\infty)$ to $\sigma^j(\overline{c_q 0 w 1})$. This is true by definition when $j = 0$, and follows for $1 \le j \le n+k+2$ since γ_{j-1} does not cross the vertical centre line C of S.

Thus γ_{n+k+2} is the concatenation of the vertical arc from $\overline{0 c_q 0 w}$ to ${}^\infty(1 c_q 0 w) \cdot (0 c_q 0 w)^\infty$ and the horizontal arc from ${}^\infty(1 c_q 0 w) \cdot (0 c_q 0 w)^\infty$ to $\overline{1 c_q 0 w}$ (which crosses C). Hence (see Figure 8) γ_{n+k+3} is the concatenation of

i) the vertical arc from $p_1 = \overline{c_q 0 w 0}$ to ${}^\infty(1 c_q 0 w)\, 0 \cdot (c_q 0 w 0)^\infty$;

ii) the horizontal arc from ${}^\infty(1 c_q 0 w)\, 0 \cdot (c_q 0 w 0)^\infty$ to the right hand edge of S;

iii) a semicircular arc outside of S; and

iv) the horizontal arc from the right hand edge of S to $\overline{c_q 0 w 1} = p_2$.

Condition a) of Definition 33 thus follows (with $\alpha(b) = {}^\infty(1 c_q 0 w)\, 0 \cdot (c_q 0 w 0)^\infty$) and conditions b) and c) are satisfied because p_1 and p_2 lie to the right of all other points in both of their orbits, and the arcs γ_j are contained in S for $1 \le j \le n+k+2$. A point $x = b \cdot f$ of $\Omega(F)$ lies in $\operatorname{Int} C_q$ if and only if both $f \succ (c_q 0 w 0)^\infty$, and $0(\hat{w} 0 c_q 1)^\infty \prec b \prec (1 \hat{w} 0 c_q)^\infty$: the latter condition is equivalent to $\sigma(b) \succ (\hat{w} 0 c_q 1)^\infty$. □

The corresponding result for the arc δ is analogous, and its proof is omitted.

Lemma 40. *The arc δ defines a disk D_q under F. A point $x = b \cdot f$ of $\Omega(F)$ lies in* $\operatorname{Int} D_q$ *if and only if*

a) $f \succ (c_q 1 w 1)^\infty$*, and*

b) $\sigma(b) \succ (\hat{w} 1 c_q 0)^\infty$.

For the purposes of the current argument, denote the arc γ, which depends on q and on the decoration w, by γ_q^w. Now $q < q_w = q_{\hat{w}}$ (Lemma 7), and hence, applying Lemma 39 with decoration $\hat{w}$, the arc $\gamma_q^{\hat{w}}$ (which joins $\overline{c_q 0 \hat{w} 0}$ to $\overline{c_q 0 \hat{w} 1}$) defines a disk $C_q^{\hat{w}}$ under F, and a point $x = b \cdot f$ of $\Omega(F)$ lies in $\operatorname{Int} C_q^{\hat{w}}$ if and only if

a) $f \succ (c_q 0 \hat{w} 0)^\infty$, and
b) $\sigma(b) \succ (w 0 c_q 1)^\infty$.

Applying the involution ϕ of Section 2.3, the arc $\phi(\gamma_q^{\hat{w}})$ (which joins $\overline{0 w 0 c_q}$ to $\overline{1 w 0 c_q}$) defines a disk $\phi(C_q^{\hat{w}})$ under F^{-1}, and a point $x = b \cdot f$ of $\Omega(F)$ lies in $\operatorname{Int} \phi(C_q^{\hat{w}})$ if and only if

a) $b \succ (c_q 0 \hat{w} 0)^\infty$, and
b) $\sigma(f) \succ (w 0 c_q 1)^\infty$.

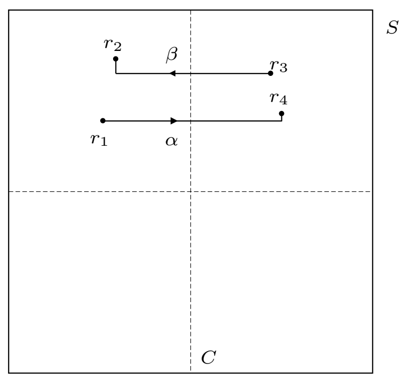

FIGURE 9. The arcs α and β

Write $\alpha = \alpha_q^w = \phi(\gamma_q^{\hat{w}})$, and similarly $\beta = \beta_q^w = \phi(\delta_q^{\hat{w}})$ (see Figure 9). Thus α joins the highest point $r_1 = \overline{0 w 0 c_q}$ on the orbit of p_1 to the highest point $r_4 = \overline{1 w 0 c_q}$ on the orbit of p_4; while β joins the highest point $r_2 = \overline{0 w 1 c_q}$ on the orbit of p_2 to the highest point $r_3 = \overline{1 w 1 c_q}$ on the orbit of p_3, and the argument above gives:

Lemma 41. *The arc α defines a disk A_q under F^{-1}. A point $x = b \cdot f$ of $\Omega(F)$ lies in $\operatorname{Int} A_q$ if and only if*

a) $b \succ (c_q 0 \hat{w} 0)^\infty$, and
b) $\sigma(f) \succ (w 0 c_q 1)^\infty$.

Lemma 42. *The arc β defines a disk B_q under F^{-1}. A point $x = b \cdot f$ of $\Omega(F)$ lies in $\operatorname{Int} B_q$ if and only if*

a) $b \succ (c_q 1 \hat{w} 1)^\infty$, and
b) $\sigma(f) \succ (w 1 c_q 0)^\infty$.

Figure 10 depicts the disks $A_{1/3}$, $B_{1/3}$, $C_{1/3}$, and $D_{1/3}$ for $w = 11$, together with the four periodic orbits of height $1/3$ and decoration w. This figure is drawn to scale, and short segments of stable and unstable manifolds cannot be discerned on

it. The disks $A_{1/3}$ and $C_{1/3}$ are shaded lightly, and the disks $B_{1/3}$ and $D_{1/3}$ are shaded heavily.

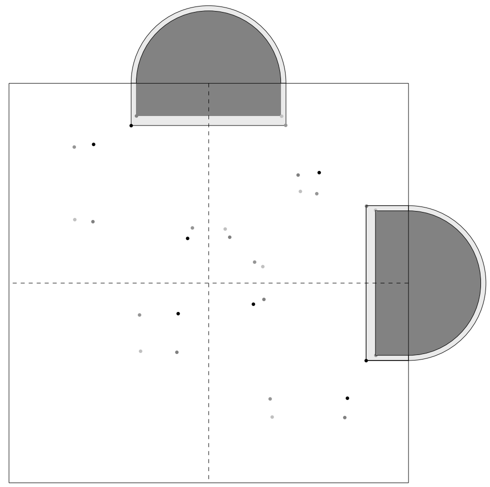

FIGURE 10. The disks $A_{1/3}$, $B_{1/3}$, $C_{1/3}$, and $D_{1/3}$ for $w = 11$

This figure motivates the following straightforward lemma.

Lemma 43. *For any decoration w and any $q \in (0, q_w)$, $D_q \subseteq C_q$ and $B_q \subseteq A_q$.*

Proof. Let $x = b\cdot f \in \Omega(F)$. It suffices to show that if $x \in \mathrm{Int}(D_q)$ then $x \in \mathrm{Int}(C_q)$. That $B_q \subseteq A_q$ will then follow since $A_q = \phi(C_q^{\hat{w}})$ and $B_q = \phi(D_q^{\hat{w}})$.

Suppose $x \in \mathrm{Int}(D_q)$: then $f \succ (c_q 1 w 1)^\infty$ and $\sigma(b) \succ (\hat{w} 1 c_q 0)^\infty$ by Lemma 40. Since both c_q and $\hat{w}$ are even words, it follows that $f \succ (c_q 0 w 0)^\infty$ and $\sigma(b) \succ (\hat{w} 0 c_q 1)^\infty$, so that $x \in \mathrm{Int}(C_q)$ by Lemma 39. $\square$

5.2. **Linking properties.** Let $q \in (0, q_w) \cap \mathbb{Q}$, and for $1 \le i \le 4$ denote by $P_{i,q}$ the periodic orbit containing the point p_i of Section 5.1. Fix a suspension $\{f_t\}$ of the horseshoe map F. Given a horseshoe periodic orbit R distinct from each $P_{i,q}$, denote by $L_{i,q}(R)$ the linking number of $P_{i,q}$ about R with respect to the suspension $\{f_t\}$.

Lemma 44.

$$\begin{aligned} L_{4,q}(R) &= L_{1,q}(R) + |R \cap A_q| \\ L_{3,q}(R) &= L_{2,q}(R) + |R \cap B_q| \\ L_{2,q}(R) &= L_{1,q}(R) + |R \cap C_q| \\ L_{3,q}(R) &= L_{4,q}(R) + |R \cap D_q|. \end{aligned}$$

In particular,

$$\begin{aligned} \max(L_{1,q}(R), L_{2,q}(R), L_{3,q}(R), L_{4,q}(R)) &= L_{3,q}(R) \qquad \textit{and} \\ \min(L_{1,q}(R), L_{2,q}(R), L_{3,q}(R), L_{4,q}(R)) &= L_{1,q}(R). \end{aligned}$$

Proof. Note first that the boundaries of the disks A_q, B_q, C_q, and D_q are composed of segments of the stable and unstable manifolds of points of the orbits $P_{i,q}$, and so cannot intersect the orbit R.

Consider the arc γ from p_1 to p_2. The curve $\gamma \cdot (F^{n+k+3}(\gamma))^{-1}$ goes clockwise around the boundary of C_q. Hence $L_{1,q}(R) = L_{2,q}(R) - |R \cap C_q|$ by Lemma 22 as required. Similarly $L_{3,q}(R) = L_{4,q}(R) + |R \cap D_q|$ (the curve $\delta \cdot (F^{n+k+3}(\delta))^{-1}$ goes anti-clockwise around the boundary of D_q).

Let $\tilde{\alpha}_q = F^{-(n+k+3)}(\alpha_q)$, an arc from r_1 to r_4. Then $\tilde{\alpha}_q \cdot \alpha_q^{-1}$ goes clockwise around the boundary of A_q, so that $L_{1,q}(R) = L_{4,q}(R) - |R \cap A_q|$ as required. Similarly $L_{3,q}(R) = L_{2,q}(R) + |R \cap B_q|$. □

Corollary 45. *Let R be any horseshoe periodic orbit distinct from each $P_{i,q}$. Then*

$$|R \cap C_q| + |R \cap B_q| = |R \cap D_q| + |R \cap A_q|.$$

Proof. Both are equal to $L_{3,q}(R) - L_{1,q}(R)$. □

5.3. **Forcing conditions.** The first main result is that $\{P_q^w : q \in \mathbb{Q}^w\}$ is totally ordered by the forcing relation. The following lemma will be used:

Lemma 46. *Let $q, q' \in (0, q_w) \cap \mathbb{Q}$ with $q' < q$. Let R be the periodic orbit of code $c_{q'}0w0$. Then $|R \cap A_q| = |R \cap C_q| = 1$, and $|R \cap B_q| = |R \cap D_q| = 0$.*

Proof. Observe first that a point of R lies in one of the disks if and only if it lies in its interior, since R is distinct from the orbits $P_{i,q}$.

a) The rightmost point $x = b \cdot f$ of R lies in $C_q \setminus D_q$. For $f = (c_{q'}0w0)^\infty$ and $b = (0\hat{w}0c_{q'})^\infty$. Now $q' = q(f) < q = q((c_q0w0)^\infty)$ and hence $f \succ (c_q0w0)^\infty$, so condition a) of Lemma 39 (for a point to lie in C_q) is satisfied.

   Since $q_w = q_{\hat{w}}$ (Lemma 7) and $q' < q < q_w$, the words $c_{q'}0\hat{w}0$ and $c_q1\hat{w}0$ are codes of periodic orbits of heights q' and q by Lemma 6. Therefore $(c_{q'}0\hat{w}0)^\infty \succ (c_q1\hat{w}0)^\infty$. Prepending the even word $\hat{w}0$ to both sides gives $\sigma(b) \succ (\hat{w}0c_q1)^\infty$, so that condition b) of Lemma 39 is satisfied.

   On the other hand, $\sigma(b) = \hat{w}0\ldots \prec (\hat{w}1c_q0)^\infty$ (since $\hat{w}$ is even), and hence condition b) of Lemma 40 (for a point to lie in D_q) is not satisfied.

b) No other point $x = b \cdot f$ of R lies in C_q. Certainly this is true if f begins in w or at the final 1 of c_q': for

$$q_w = \min_{0 \le i \le k+2} q\left(\sigma^i((10w0)^\infty)\right)$$

   gives that $q(f) \ge q_w > q = q((c_q0w0)^\infty)$ for any $f \in \{0,1\}^{\mathbb{N}}$ which starts either with $10w010$, or with a final subword of w followed by 010, so that $f \prec (c_q0w0)^\infty$, and condition a) of Lemma 39 is not satisfied. Thus condition a) of this lemma can only be satisfied if

$$\begin{aligned} f &= 10^{\kappa_i}1^2\ldots 1^2 0^{\kappa_{m'}}10w0\ldots \qquad \text{and} \\ b &= 10^{\kappa_{i-1}}1^20^{\kappa_{i-2}}1^2\ldots 1^20^{\kappa_1}10\hat{w}0\ldots \qquad \text{for some } i \text{ with } 1 < i \le m' \end{aligned}$$

(where $c_q' = 10^{\kappa_1}1^2 \dots 1^2 0^{\kappa_{m'}}1$). Hence

$$q(10\sigma(b)) \le q' < q_w = q_{\hat{w}} \le q\left(10(\hat{w}0c_q1)^\infty\right)$$

(the first inequality coming from Lemma 5 and the fact that $c_{q'}$ is palindromic, and the second from the above formula for q_w), so that $10\sigma(b) \succ 10(\hat{w}0c_q1)^\infty$, i.e. $\sigma(b) \prec (\hat{w}0c_q1)^\infty$, and condition b) of Lemma 39 fails.

Hence $|R \cap C_q| = 1$ and $|R \cap D_q| = 0$. So $|\widehat{R} \cap C_q^{\hat{w}}| = 1$ and $|\widehat{R} \cap D_q^{\hat{w}}| = 0$, and therefore $|R \cap A_q| = |\phi(\widehat{R}) \cap \phi(C_q^{\hat{w}})| = 1$ and $|R \cap B_q| = |\phi(\widehat{R}) \cap \phi(D_q^{\hat{w}})| = 0$ as required. $\square$

Theorem 19 a) follows immediately from the following result.

Theorem 47. *Let $q \in \mathbb{Q}^w$ and $q' \in (0,q) \cap \mathbb{Q}$. Then $P^w_{q'} \ge P^w_q$.*

Proof. Let R be the periodic orbit of code $c_{q'}0w0$. Lemmas 44 and 46 give that

$$L_{2,q}(R) = L_{3,q}(R) = L_{4,q}(R) = L_{1,q}(R) + 1.$$

Suppose first that $L_{1,q}(R)$ is not divisible by $n+k+3$. It will be shown that $(p_1\,;F)$ is unremovable in (D^2, R), which will establish the result. Note first that $(p_1\,;F)$ is certainly separated from R, since R and $P_{1,q}$ have different braid types (Remark 31). To show that $(p_1\,;F)$ is uncollapsible, suppose that $g_j \to g$ is a sequence of homeomorphisms $D^2 \to D^2$, and $r_j \to r$ in D^2 is such that r_j is a period $n+k+3$ point of g_j with $(r_j\,;g_j) \sim (p_1\,;F)$ for each j. The fact that $n+k+3 \not| L_{1,q}$ means (by Lemma 23) that r cannot be a fixed point of g. If r were a period $(n+k+3)/\ell$ point of g for some ℓ with $1<\ell<n+k+3$, then for j sufficiently large the braid type of the g_j-orbit of r_j would be reducible with $(n+k+3)/\ell$ reducing curves, contradicting the fact that $P_{1,q}$ has pseudo-Anosov braid type. Thus r is a period $n+k+3$ point of g, establishing the uncollapsibility of $(p_1\,;F)$. Finally, to show that $(p_1\,;F)$ is essential: if $r \in \operatorname{snc}(p_1\,;F)$, then r lies on a periodic orbit of braid type $\operatorname{bt}(P_{1,q})$ and has linking number $L_{1,q}(R)$ about R with respect to some suspension of F by Lemma 30. Since w is a lone decoration and $L_{2,q}(R) = L_{3,q}(R) = L_{4,q}(R) = L_{1,q}(R)+1$, it follows that $r \in P_{1,q}$, and hence $I(p_1\,;F) = -|\operatorname{snc}(p_1\,;F)| \neq 0$, so $(p_1\,;F)$ is essential as required.

On the other hand, if $L_{1,q}(R)$ is divisible by $n+k+3$, then $L_{2,q}(R)$, $L_{3,q}(R)$, and $L_{4,q}(R)$ are not. The above argument can then be repeated with p_2, the only part which is any different being the proof that $I(p_2\,;F) \neq 0$. For this part, observe first that by Lemma 30 the only points which can lie in $\operatorname{snc}(p_2\,;F)$ are points of the form $F^i(p_j)$ for $j \in \{2,3,4\}$, since these are the only periodic points of F which lie on periodic orbits of braid type $\operatorname{bt}(P_{2,q})$ and have linking number $L_{2,q}(R)$ about R. However, since $|R \cap B_q| = |R \cap D_q| = 0$, Theorem 35 gives that $(p_2\,;F) \sim (p_3\,;F) \sim (p_4\,;F)$. Hence if $F^i(p_j) \in \operatorname{snc}(p_2\,;F)$ for some $j \in \{2,3,4\}$, then (Lemma 32) this is true for all $j \in \{2,3,4\}$. Since points of $P_{2,q}$ and $P_{4,q}$ have index $+1$, while

those of $P_{3,q}$ have index -1, it follows that $I(p_2\,;\,F) = |\operatorname{snc}(p_2\,;\,F)|/3 > 0$ as required. $\square$

Remark 48. A cleaner approach to this proof would be to start by using the fact that $R \cap D_q = \emptyset$ to prune away the periodic orbits $P_{3,q}$ and $P_{4,q}$ by an isotopy which leaves R, $P_{1,q}$, and $P_{2,q}$ untouched (a suitable pruning disk can be constructed by analogy with Lemma 50 below). There would then remain only two periodic orbits of the braid type of $P_{1,q}$, whose linking numbers about R would differ by 1. However, in order to take this approach it is necessary to ensure that the indices of p_1 and p_2 are unchanged after the pruning isotopy, which requires more careful control of the support of this isotopy. The technical details involved in doing this are more complicated than the approach taken in the proof above.

The next lemma describes how the disk C_q can be enlarged to a pruning disk Δ_q which contains all of the points p_i. (See Figure 11.) Δ_q is "approximately the same" as C_q in the sense that if q has large denominator then the boundary of Δ_q is close to that of C_q, and in particular $\Delta_q \setminus C_q$ cannot contain periodic points of low period. This idea is encapsulated in the following definition:

Definition 49. Let $q = m/n \in \mathbb{Q}$. If $X, Y \subseteq D^2$, write $X \subseteq_q Y$ if any periodic point of F of period less than $n/2$ which lies in X also lies in Y.

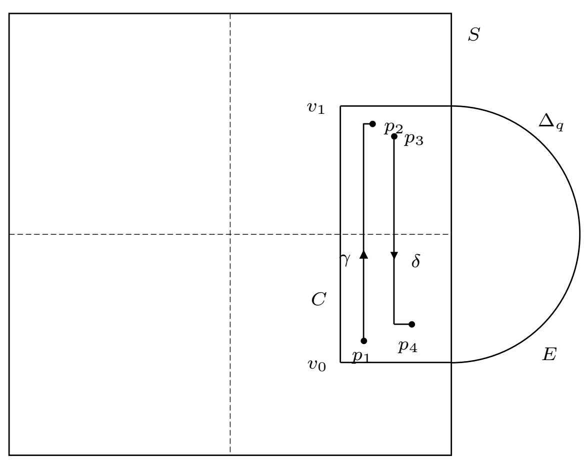

FIGURE 11. The pruning disk Δ_q

Lemma 50. *Let v_0 and v_1 be the points ${}^{\infty}0c_q0w_1^0 \cdot (c_q0w011)^{\infty}$. Then v_0 and v_1 are the vertices of a pruning disk Δ_q which contains $\{p_1, p_2, p_3, p_4\}$. Moreover $\Delta_q \subseteq_q C_q$.*

Proof. Observe that a point $x = b \cdot f$ of $\Omega(F)$ lies in $\operatorname{Int}(\Delta_q)$ if and only if $f \succ (c_q0w011)^{\infty}$ and $\sigma(b) \succ \hat{w}0c_q0^{\infty}$.

Let C and E be the segments of stable and unstable manifold constituting the boundary of Δ_q. Observe that, for $n \geq 1$, $F^n(C)$ (respectively $F^{-n}(E)$) is a segment of stable (respectively unstable) manifold, with endpoints $F^n(v_i)$ (respectively

$F^{-n}(v_i))$, which is contained in the central square S of the horseshoe. Thus to show that Δ_q is a pruning disk (Definition 36), it is enough to show that $F^n(v_i) \not\in \mathrm{Int}(\Delta_q)$ for all $n \in \mathbb{Z}$.

Now $q_{w01} = \min_{0\le i\le k+4} q(\sigma^i((10w010)^\infty)) = \min_{0\le i \le k+2} q(\sigma^i((10w0)^\infty)) = q_w$, and hence $q < q_{w01}$. Thus (Lemma 6) $c_q 0w011$ is the code of a periodic orbit, whose rightmost point is on the stable leaf containing C. It follows that $F^n(v_i) \not\in \mathrm{Int}(\Delta_q)$ for all $n \ge 0$.

If $n < 0$, then $h(\sigma^n(v_i)) \succ (c_q 0w011)^\infty$ could only occur if

$$h(\sigma^n(v_i)) = 10^{\kappa_r}1^2\dots 1^2 0^{\kappa_m} 10 w_1^0 (c_q 0w011)^\infty$$

for some $1 \le r \le m$ (recall the horiztonal and vertical coordinate functions h and v from Section 2.1). If $r > 1$ then this is not possible (as $q(10^{\kappa_r}1^2\dots 1^2 0^{\kappa_m}10\dots) > q$), while if $r = 1$ then $\sigma(v(\sigma^n(v_i))) = 0^\infty \not\succ \hat{w}0c_q0^\infty$. Thus Δ_q is a pruning disk as required.

It is straightforward that Δ_q contains all of the p_i, since $\left(c_{q1}^{\,0} w_1^0\right)^\infty \succ (c_q 0w011)^\infty$ and $\left(\hat{w}_1^0 c_{q1}^{\,0}\right)^\infty \succ \hat{w}0c_q0^\infty$.

For the final part of the lemma, suppose that $x = \overline{c}$ is a point of a period $N = |c| < n/2$ orbit R of F. It is required to show that if $x \in \mathrm{Int}(\Delta_q)$ then $x \in \mathrm{Int}(C_q)$. Suppose for a contradiction, then, that $x \in \mathrm{Int}(\Delta_q) \setminus \mathrm{Int}(C_q)$: that is, by Lemma 39, either

$$\begin{array}{ccccl} (c_q 0w011)^\infty & \prec & c^\infty & \preceq & (c_q 0w0)^\infty, \quad \text{or} \\ \hat{w}0c_q0^\infty & \prec & \sigma(\hat{c}^\infty) & \preceq & (\hat{w}0c_q1)^\infty. \end{array}$$

If the former inequalities hold, then $q(c^\infty) = q$. Since $|c| < n/2$, both c and cc must be initial words of c_q, and so $c = 10^{\kappa_1}1^2\dots 1^2 0^{\kappa_r}1$ for some $r < m/2$. Now the code c_R of R is some cyclic permutation of c, so $c_R = 10^{\kappa_i}1^2\dots 1^2 0^{\kappa_r} 1^2 0^{\kappa_1}\dots 1^2 0^{\kappa_{i-1}}1$ for some i between 1 and r. Removing the even initial subword $10^{\kappa_1}1^2\dots 1^2 0^{\kappa_{i-1}}1$ (of length s, say) from each term in the inequalities gives

$$\sigma^s((c_q 0w011)^\infty) \prec {c_R}^\infty \preceq \sigma^s((c_q 0w0)^\infty),$$

and hence $q(R) = q({c_R}^\infty) \ge q$. On the other hand, $q(R) \le q(c^\infty) = q$, and hence $q(R) = q = m/n$, and so R has period at least n. This is the required contradiction.

If the latter inequalities hold, then (removing the even initial subword $\hat{w}0$ from each term) there is some point d^∞ of R such that

$$c_q 0^\infty \prec \hat{d}^\infty \preceq (c_q 1\hat{w}0)^\infty,$$

and the proof proceeds in exactly the same way. $\square$

The description of the invariant r^w follows relatively easily from the following theorem:

Theorem 51. *Let R be a period N horseshoe orbit, and let $q = m/n \in \mathbb{Q}^w$ be such that $n > 2N$. Then $R \ge P_q^w$ if and only if $R \cap A_q \ne \emptyset$ and $R \cap C_q \ne \emptyset$.*

Proof. Notice that R cannot intersect the boundary of any of the disks A_q, B_q, C_q, D_q, since its period is less than the period $n+k+3$ of P_q^w.

If $R \cap C_q = \emptyset$, then $R \not\geq P_q^w$ by Lemma 50 and Corollary 38. Similarly, if $R \cap A_q = \emptyset$ then $\phi(R) \cap \phi(A_q) = \phi(R) \cap C_q^{\hat{w}} = \emptyset$, and $\phi(\Delta_q^{\hat{w}})$ is a pruning disk which is disjoint from R but contains the highest points $\{r_1, r_2, r_3, r_4\}$ of the orbits P_q^w.

Suppose, then, that $R \cap A_q \neq \emptyset$ and $R \cap C_q \neq \emptyset$. It is required to prove that $R \geq P_q^w$. By Lemma 44, the linking number L_1 is strictly smaller than the linking numbers L_2, L_3, and L_4. If L_1 is not divisible by $n+k+3$, then an identical argument to that in the proof of Theorem 47 shows that $(p_1\,;\,F)$ is unremovable in (D^2, R), which establishes the result. On the other hand, if L_1 is divisible by $n+k+3$ then L_2, L_3, and L_4 are not. Either one of these linking numbers is distinct from the other two, or all three are equal, and in either case the proof proceeds identically to that of Theorem 47.

□

5.4. **The invariant r^w.** The next result completes the proof of Theorem 19 for even decorations.

Theorem 52. *Let R be any horseshoe periodic orbit, and $r^w(R) \in (0, q_w] \cap \mathbb{Q}$ be as given in Definition 16. Then for $q \in \mathbb{Q}^w$*

$$0 < q < r^w(R) \quad \Longrightarrow \quad R \not\geq P_q^w \qquad \text{and}$$
$$r^w(R) < q < q_w \quad \Longrightarrow \quad R \geq P_q^w.$$

In particular, r^w is a braid type invariant.

Proof. Let $q = m/n \in \mathbb{Q}^w$. Suppose first that $0 < q < r^w(R)$: it is required to show that $R \not\geq P_q^w$. Without loss of generality, assume that $n/2$ is greater than the period of R, so that Theorem 51 applies (if not, replace q with $q' \in (q, r^w(R)) \cap \mathbb{Q}^w$ having sufficiently large denominator. Then $P_q^w \geq P_{q'}^w$ by Theorem 47, so showing that $R \not\geq P_{q'}^w$ shows that $R \not\geq P_q^w$).

Assume for a contradiction that $R \geq P_q^w$, so that $R \cap A_q \neq \emptyset$ and $R \cap C_q \neq \emptyset$ by Theorem 51.

Let $x = b \cdot f \in R \cap C_q$. By Lemma 39, $f \succ (c_q 0 w 0)^\infty$ and $\sigma(b) \succ (\hat{w} 0 c_q 1)^\infty$. If $\sigma(b) = \hat{w} 0 b'$ for some $b' \in \{0,1\}^{\mathbb{N}}$ then $b \cdot f = b' 0 w_1^0 \cdot f$, where $f \succ (c_q 0 w 0)^\infty$ (so $q(f) \leq q$) and $b' \succ (c_q 1 \hat{w} 0)^\infty$ (so $q(b') \leq q$). Thus there is a word $010 w_1^0 10$ in $\overline{c_R}$ which forces $\lambda^w(R) \leq q$.

On the other hand, if $\sigma(b)$ doesn't begin $\hat{w}0\ldots$ then $\sigma(b) = \tilde{\hat{v}} b'$, where $\hat{v}$ is a non-empty even initial subword of $\hat{w}0$. Then $b \cdot f = b' \check{v}_1^0 \cdot f$ where v is a non-empty even final subword of $0w$, and $f \succ (c_q 0 w 0)^\infty$ so that $q(f) \leq q$. Hence $\mu^w(R) \leq q$.

To summarize: the existence of a point of $R \cap C_q$ means that either $\lambda^w(R) \leq q$ or $\mu^w(R) \leq q$.

Now let $x = b \cdot f \in R \cap A_q$ (to simplify the notation the same symbols x, b, and f are used although x must be a different point of R). Thus $\phi(x) = f \cdot b \in \widehat{R} \cap C_q^{\hat{w}}$,

so that either $\lambda^{\hat{w}}(\widehat{R}) \le q$ or $\mu^{\hat{w}}(\widehat{R}) \le q$. Thus either $\lambda^w(R) \le q$ or $\nu^w(R) \le q$ by Remark 15.

So the fact that both $R \cap C_q$ and $R \cap A_q$ are non-empty means that either $\lambda^w(R) \le q$, or both $\mu^w(R)$ and $\nu^w(R)$ are less than or equal to q. This is precisely to say that $r^w(R) \le q$, which is the required contradiction.

For the converse, suppose that $r^w(R) < q < q_w$: it is required to show that $R \ge P_q^w$. Once again, assume without loss of generality that $n/2$ is greater than the period of R (if not, replace q with an element of $(r^w(R), q)$ having sufficiently large denominator).

Since $q > r^w(R) = \min(\lambda^w(R), \max(\mu^w(R), \nu^w(R)))$, at least one of the following two possibilities holds: that $q > \lambda^w(R)$, or that q is greater than both $\mu^w(R)$ and $\nu^w(R)$. It will be shown that in either case R contains points of both A_q and C_q, so that $R \ge P_q^w$ by Theorem 51 as required.

a) Suppose that $q > \lambda^w(R)$. Thus there is a point $x = b \cdot f$ of R with $q(f) < q$ and $b = {}^0_1\hat{w}{}^0_1 b'$, where $q(b') < q$. In particular, $f \succ (c_q 0 w 0)^\infty$ and $\sigma(b) = \hat{w}{}^0_1 b' \succ (\hat{w} 0 c_q 1)^\infty$, so $x \in C_q$ by Lemma 39. Similarly, $q > \lambda^{\hat{w}}(\widehat{R}) = \lambda^w(R)$, so there is a point of $\widehat{R}$ in $C_q^{\hat{w}}$, and hence a point of R in A_q.
b) Suppose that $q > \mu^w(R)$ and $q > \nu^w(R)$. The fact that $q > \mu^w(R)$ means that there is a point $x = b \cdot f$ of R with $q(f) < q$ and $b = {}^0_1\hat{\check{v}}\ldots$, where v is a non-empty even final subword of $0w$. Thus $f \succ (c_q 0 w 0)^\infty$ and $\sigma(b) \succ (\hat{w} 0 c_q 1)^\infty$, so $x \in C_q$ by Lemma 39. Similarly, $q > \nu^w(R) = \mu^{\hat{w}}(\widehat{R})$ means that there is a point of $\widehat{R}$ in $C_q^{\hat{w}}$, and hence a point of R in A_q.

□

5.5. **The case of odd decoration.** The case where w is an odd decoration works similarly, the only substantial changes being the exchange of the rôles of C_q and D_q (and similarly of A_q and B_q), and the replacement of the inclusions of Lemma 43 with “approximate inclusions”.

Define the points p_i and the arcs γ, δ, α, and β exactly as in Section 5.1: observe that now $\operatorname{index}(p_i, F^{n+k+3}) = (-1)^{i+1}$ rather than $(-1)^i$ as before. Lemmas 39−42 (describing the disks defined by these arcs) and Lemma 44 and Corollary 45 (describing the linking numbers of the orbits $P_{i,q}$ about a periodic orbit R) are unchanged. However Lemma 43 is false, as can clearly be seen from Figure 12, which is a schematic depiction of the points p_i (with their correct relative horizontal and vertical ordering), the arcs γ and δ, and the images of these arcs under F^{n+k+3}. What is true, though, is that the disk C_q is *approximately* contained in D_q in the sense of Definition 49.

Lemma 43 (odd version). — *For any odd decoration w and any $q \in (0, q_w)$, $C_q \subseteq_q D_q$ and $A_q \subseteq_q B_q$.*

Proof. Let $q = m/n$. Let $x = \overline{c}$ be a point of a period $N = |c| < n/2$ orbit R of F. It suffices to show that if $x \in \operatorname{Int}(C_q)$ then $x \in \operatorname{Int}(D_q)$. The result will then follow

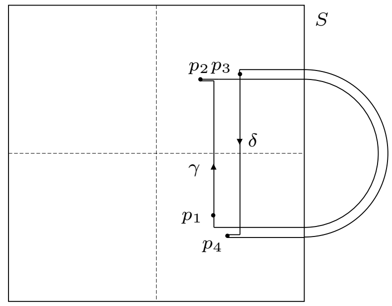

FIGURE 12. The disks C_q and D_q when w is odd

since $A_q = \phi(C_q^{\hat{w}})$ and $B_q = \phi(D_q^{\hat{w}})$. The proof works in exactly the same way as the final part of the proof of Lemma 50.

□

Only a minor modification is needed to Lemma 46:

Lemma 46 (odd version). — *Let $q, q' \in (0, q_w) \cap \mathbb{Q}$ with $q' < q$. Let R be the periodic orbit of code $c_{q'}0w0$. Then $|R \cap A_q| = |R \cap C_q| = 0$, and $|R \cap B_q| = |R \cap D_q| = 1$.*

The proof works in exactly the same way, the only small difference being the need to show that points of R other than the rightmost point lie in *neither* C_q nor D_q: this uses the fact that if w is odd then $q(10w010\ldots) = q(10w110\ldots)$.

Theorem 47 is true as stated for odd decorations, and the proof is identical except that $L_{3,q}(R)$ plays the rôle of $L_{1,q}(R)$, since the revised Lemma 46 gives that

$$L_{1,q}(R) = L_{2,q}(R) = L_{4,q}(R) = L_{3,q}(R) - 1.$$

The pruning disk which contains all of the points p_i is different from that used in the case of even decoration.

Lemma 50 (odd version). — *Let v_0 and v_1 be the points ${}^{\infty}0c_q1w_1^0 \cdot (c_q0w10)^{\infty}$ if w ends with the word 01^{2i} for some $i \geq 0$, and the points ${}^{\infty}0c_q1w_1^0 \cdot (c_q0w110)^{\infty}$ otherwise. Then v_0 and v_1 are the vertices of a pruning disk Δ_q which contains $\{p_1, p_2, p_3, p_4\}$. Moreover $\Delta_q \subseteq_q D_q$.*

The only change to Theorem 51 is the replacement of A_q and C_q with B_q and D_q. The proof works identically.

Theorem 51 (odd version). — *Let R be a period N horseshoe orbit, and suppose that $q = m/n \in \mathbb{Q}^w$ is such that $n > 2N$. Then $R \geq P_q^w$ if and only if $R \cap B_q \neq \emptyset$ and $R \cap D_q \neq \emptyset$.*

Finally, the statement of Theorem 52 is unchanged, and its proof works in just the same way. Using B_q and D_q in place of A_q and C_q is exactly what is required to compensate for the changes introduced because w is odd, and because ${}^{+}w = 1w$ and $w^{+} = w1$, rather than $0w$ and $w0$ as in the even case.

6. Examples and applications

6.1. **Generalities.** The following straightforward lemma will be useful in this section.

Lemma 53. *Let w be a lone decoration.*

a) If R is a horseshoe periodic orbit of decoration w, then $r^w(R) = q(R)$.

b) If R is any horseshoe periodic orbit and $r^w(R) \neq q_w$, then $r^w(R) \geq q(R)$.

c) If R is any horseshoe periodic orbit, then $r^w(R) = r^{\hat{w}}(\widehat{R})$.

Proof. a) is immediate from Theorem 47, Theorem 52, and Lemma 11 c), while c) is immediate from Remark 15.

For b), suppose that $r^w(R) < q(R)$. Pick $q_1, q_2 \in (r^w(R), \min(q(R), q_w)) \cap \mathbb{Q}^w$ with $q_1 < q_2$. Since $q_2 < q(R)$, Lemma 11 b) gives $P^*_{q_2} \geq R$. Since $r^w(R) < q_1$, Theorem 52 gives $R \geq P^w_{q_1}$. Hence $P^*_{q_2} \geq P^w_{q_1}$, and Lemma 11 c) gives $q_2 = q(P^*_{q_2}) \leq q(P^w_{q_1}) = q_1$, which is the required contradiction. □

6.2. **Star decorations.** For each rational $m/n \in (0, 1/2)$, consider the "star" decoration $w_{m/n}$ which is defined by removing the initial symbols 10 and the final symbols 01 from the word $c_{m/n}$ of Definition 1. Thus, using the notation of Lemma 5,

$$w_{m/n} = 0^{\kappa_1 - 1} 1^2 0^{\kappa_2} 1^2 0^{\kappa_3} 1^2 \ldots 1^2 0^{\kappa_{m-1}} 1^2 0^{\kappa_m - 1}.$$

These decorations were considered in [dCH04]: their name is due to the fact that the train tracks for periodic orbits with decoration $w_{m/n}$ are all star-shaped, with n branches: the pseudo-Anosov has an n-pronged singularity corresponding to the vertex of the star, whose prongs are rotated by m/n.

The scope $q_{w_{m/n}}$ of the decoration $w_{m/n}$ is m/n. In order to simplify the notation, the periodic orbits $P^{w_{m/n}}_q$ of decoration $w_{m/n}$ and height $q \in (0, m/n)$ will be denoted $P^{m/n}_q$ and the invariants $r^{w_{m/n}}$ will be denoted $r^{m/n}$ throughout this subsection.

It is shown in Lemma 17 and Corollary 18 of [dCH04] that each $w_{m/n}$ is a lone decoration, and that the periodic orbits $P^{m/n}_q$ all have pseudo-Anosov braid type. Thus for each fixed m/n the set

$$\mathcal{D}^{m/n} = \{\mathrm{bt}(P^{m/n}_q, F) \,:\, q \in (0, m/n)\}$$

is totally ordered by forcing, with $P^{m/n}_{q'} \geq P^{m/n}_q$ if and only if $q' \leq q$ (Theorem 47). The aim in this section is to use Theorem 52 to determine the forcing between braid types in distinct families $\mathcal{D}^{m/n}$: that is, for each m/n, m'/n', q, and q', to determine whether or not $P^{m'/n'}_{q'} \geq P^{m/n}_q$. This completes the proof of Theorem 15 d) of [dCH04].

Recall that the *rotation number* $\rho(R) \in (0, 1/2]$ of a horseshoe periodic orbit R (other than a fixed point) is its rotation number in the annulus obtained by puncturing the disk at the fixed point of code 1. The *rotation interval* $\rho i(R) \subseteq (0, 1/2]$

is the set of rotation numbers of periodic orbits forced by R. Clearly $\rho i(R)$ is a braid type invariant, and if $R \geq S$ then $\rho i(R) \supseteq \rho i(S)$.

Lemma 17 of [dCH04] states that $\rho i(P_q^{m/n}) = [q, m/n]$. Hence if $P_{q'}^{m'/n'} \geq P_q^{m/n}$ then $[q, m/n] \subseteq [q', m'/n']$. The main result of this section is a near-converse to this statement.

Theorem 54. *Let $m/n, m'/n', q' \in (0, 1/2) \cap \mathbb{Q}$ with $q' < m'/n'$. Then*

$$r^{m/n}(P_{q'}^{m'/n'}) = \begin{cases} q' & \text{if } q' < \frac{m}{n} \leq \frac{m'}{n'} \\ \frac{m}{n} & \text{otherwise.} \end{cases}$$

That is, if $q' \neq q$ then $P_{q'}^{m'/n'} \geq P_q^{m/n}$ if and only if $[q, m/n] \subseteq [q', m'/n']$. In the language of Conjecture 8, this means that $w_{m/n} \preceq w_{m'/n'}$ if and only if $m/n \leq m'/n'$.

Proof. If $m/n \leq q'$ or $m/n > m'/n'$, then $\rho i(P_{q'}^{m'/n'}) = [q', m'/n']$ does not contain the rotation interval $[q, m/n]$ of $P_q^{m/n}$ for any q: that is, $P_{q'}^{m'/n'}$ does not force any $P_q^{m/n}$, and hence $r^{m/n}(P_{q'}^{m'/n'}) = m/n$ as required.

If $m'/n' = m/n$, then $r^{m/n}(P_{q'}^{m'/n'}) = q'$ as required by Lemma 53 a).

Suppose, then, that $q' < m/n < m'/n'$. Lemma 53 b) gives that $r^{m/n}(P_{q'}^{m'/n'}) \geq q'$, so it suffices to show that $r^{m/n}(P_{q'}^{m'/n'}) \leq q'$. To do this, it is enough to show that $\mu^{w_{m/n}}(P_{q'}^{m'/n'}) \leq q'$ and $\nu^{w_{m/n}}(P_{q'}^{m'/n'}) \leq q'$.

Let $R = P_{q'}^{m'/n'}$ with $c_R = c_{q'} 0 w_{m'/n'} 0$, and write

$$w_{m'/n'} = 0^{\ell_1 - 1} 1^2 0^{\ell_2} 1^2 0^{\ell_3} 1^2 \ldots 1^2 0^{\ell_{m'-1}} 1^2 0^{\ell_{m'} - 1}.$$

Similarly, write

$$w_{m/n} = 0^{\kappa_1 - 1} 1^2 0^{\kappa_2} 1^2 0^{\kappa_3} 1^2 \ldots 1^2 0^{\kappa_{m-1}} 1^2 0^{\kappa_m - 1}.$$

Consider the point of R with itinerary $\sigma^{n'+1}(\overline{c_R}) = \overline{0 w_{m'/n'} 0 c_{q'}} = b \cdot f$, so that $f = \left(0 w_{m'/n'} 0 c_{q'}\right)^\infty$ and $b = \left(c_{q'} 0 w_{m'/n'} 0\right)^\infty$ (the latter using the fact that $w_{m'/n'}$ and $c_{q'}$ are palindromic). It will be shown that $f = 0 \tilde{v} f'$ for some $f' \in \{0, 1\}^{\mathbb{N}}$, where v is a non-empty even initial subword of $w^+_{m/n}$. Thus

$$\sigma^{n' + |v|}(\overline{c_R}) = b 0 \tilde{v} \cdot f',$$

for v a non-empty even initial subword of $w^+_{m/n}$, so $\nu^{w_{m/n}}(P_{q'}^{m'/n'}) \leq q(b) = q'$. Then

$$\mu^{w_{m/n}}(P_{q'}^{m'/n'}) = \nu^{\hat{w}_{m/n}}(\hat{P}_{q'}^{m'/n'}) = \nu^{w_{m/n}}(P_{q'}^{m'/n'}) \leq q',$$

so that $r^{m/n}(P_{q'}^{m'/n'}) \leq q'$ as required.

Thus it only remains to establish the claim, that $f = \left(0 w_{m'/n'} 0 c_{q'}\right)^\infty$ is of the form $0 \tilde{v} f'$, where v is a non-empty even initial subword of $w^+_{m/n} = w_{m/n} 0$.

Observe first that the words

$$\begin{aligned} w_{m'/n'}0c_{q'} &= 0^{\ell_1-1}1^20^{\ell_2}1^2\ldots1^20^{\ell_{m'-1}}1^20^{\ell_{m'}}10\ldots \quad \text{and} \\ w_{m/n}0 &= 0^{\kappa_1-1}1^20^{\kappa_2}1^2\ldots1^20^{\kappa_{m-1}}1^20^{\kappa_m} \end{aligned}$$

must disagree before the shorter of their lengths. For

a) There is a subword of the form 010 in $w_{m'/n'}0c_{q'}$ but no such subword in $w_{m/n}0$. Hence if $|w_{m/n}0| \geq |w_{m'/n'}0c_{q'}|$ then the two words must disagree.

b) On the other hand, if $|w_{m/n}0| < |w_{m'/n'}0c_{q'}|$ and the words do not disagree, then $w_{m'/n'}0c_{q'} = w_{m/n}0\ldots$, and so $10w_{m'/n'}0c_{q'} = 10w_{m/n}0\ldots$ and (since $10w_{m/n}0$ is an odd word)

$$10w_{m'/n'}0c_{q'} \succ 10w_{m/n}010^\infty.$$

Taking the height of both sides gives $m'/n' \leq m/n$, a contradiction.

Since $m'/n' = q(10w_{m'/n'}0c_{q'}\ldots) > q(10w_{m/n}010\ldots) = m/n$, the word $w_{m'/n'}0c_{q'}$ is greater than $w_{m/n}0$ in the unimodal order. Let u be the longest initial word on which they agree, and v be the length $|u|+1$ initial subword of $w_{m/n}0$. Then either u is even and $v = u0$, or u is odd and $v = u1$. In either case, v is a non-trivial even initial subword of $w^+_{m/n} = w_{m/n}0$ and $0\tilde{v}$ is an initial subword of $0w_{m'/n'}0c_{q'}$ as required. □

6.3. **Decorations of the form 1^{2i+1}.** For each $i \geq 0$, consider the decoration $w_i = 1^{2i+1}$, with scope $q_{w_i} = q\left(\left(101^{2i+1}0\right)^\infty\right) = 1/2$. It will be shown that w_i is lone for all i, yielding corresponding braid type invariants r^{w_i} by Theorem 19. These invariants will then be used to show that $P_q^{w_i}$ has pseudo-Anosov braid type for all i and all $q \in (0, 1/2)$.

In order to simplify the notation, the periodic orbits $P_q^{w_i}$ will be denoted P_q^i and the invariants r^{w_i} will be denoted r^i throughout this subsection. Similarly, λ^{w_i}, μ^{w_i} and ν^{w_i} will be abbreviated to λ^i, μ^i, and ν^i.

Remark 55. Decorations of the form 1^{2i} ($i \geq 0$) are also lone: 1^{2i} is the decoration $w_{(i+1)/(2i+3)}$ of Section 6.2.

The proof that the decorations w_i are lone is by induction on i: the fact that w_i is lone will be established by using the fact that w_{i-1} is lone, and hence that r^{i-1} is a braid type invariant. The proof will also make use of the following theorem and lemma, which can be found in [Hal94], Theorem 56 appearing as Theorems 3.11 and 3.15, and Lemma 57 appearing as Lemma 3.3.

Theorem 56. *The rotation interval of any horseshoe periodic orbit R is of the form*

$$\rho i(R) = [q(R), \mathrm{rhe}(R)] \cap \mathbb{Q},$$

where $\mathrm{rhe}(R) \in [q(R), 1/2] \cap \mathbb{Q}$ *is equal to* $1/2$ *if and only if* $\overline{c_R}$ *contains either the word* 01010 *or the word* $01^{2m+1}0$ *for some* $m \geq 1$.

Lemma 57. *Suppose that $c \in \{0,1\}^{\mathbb{N}}$ has $q(c) = 1/2$. Then $c = 0\dots$, or $c = 11\dots$, or $c = 101^{2k-1}0\dots$ for some $k \geq 1$.*

The following lemma will also be used in the proof.

Lemma 58. *Let $i \geq 0$, and let R be a horseshoe periodic orbit with $r^i(R) < 1/2$. Then $\overline{c_R}$ contains a word of the form $01^{2k+1}0$ for some $k \leq i+2$.*

Proof. (Note that it is not assumed in this proof that r^i is a braid type invariant.)

Since $r^i(R) = \min(\lambda^i(R), \max(\mu^i(R), \nu^i(R))) < 1/2$, either $\lambda^i(R) < 1/2$ or $\nu^i(R) < 1/2$.

If $\lambda^i(R) < 1/2$, then there is a word $v = {}^0_1 w^i {}^0_1$ such that some shift of $\overline{c_R}$ is of the form $bv \cdot f$, where $q(b) < 1/2$ and $q(f) < 1/2$ (i.e. $b = 10\dots$ and $f = 10\dots$). Thus $bv \cdot f = \dots 01^0_1 1^{2i+1} {}^0_1 10\dots$, which contains a block of 1s of odd length either $2i+1$, $2i+3$, or $2i+5$ as required.

Similarly, if $\nu^i(R) < 1/2$ then there is some non-empty even final subword $v = 1^{2j}$ $(1 \leq j \leq i+1)$ of $w^{i+} = 1^{2i+2}$ with the property that some shift of $\overline{c_R}$ is of the form $b^0_1 \tilde{v} \cdot f = b^0_1 1^{2j-1} 0 \cdot f$, where $q(b) < 1/2$ and so $b = 10\dots$. Hence $\overline{c_R}$ contains a block of 1s of odd length either $2j-1$ or $2j+1 \leq 2i+3$ as required. □

Theorem 59. *For each $i \geq 0$, the decoration $w_i = 1^{2i+1}$ is lone.*

Proof. The proof is by induction on i. For $i = 0$, it is required to show that for each $q = m/n \in (0,1/2) \cap \mathbb{Q}$, the four horseshoe periodic orbits $P_q^{w_0}$ of codes $c_q {}^0_1 1^0_1$ are the only horseshoe periodic orbits of their braid types. Now the only other horseshoe periodic orbits of height q and period $n+4$ are those of decoration 0, i.e. those with codes $c_q {}^0_1 0^0_1$ (since $q_0 = 1/3$, there is in fact nothing to prove if $q > 1/3$). However $\rho i(P_q^{w_0}) = [q, 1/2]$ by Theorem 56 (since $\overline{c_q {}^0_1 1^0_1}$ contains the word $01^0_1 1^0_1 10$), while the periodic orbits of codes $c_q {}^0_1 0^0_1$ have rotation intervals with right hand endpoints less than $1/2$, since $\overline{c_q {}^0_1 0^0_1}$ contains neither of the words of Theorem 56. (In fact these are periodic orbits with star decoration $w_{1/3}$, and hence their rotation intervals have right hand endpoint $1/3$.)

Now let $i > 0$. By the inductive hypothesis, r^{i-1} is a braid type invariant.

Lemma 58 gives $r^{i-1}(P_q^i) = 1/2$. For writing $R = P_q^i$, with

$$c_R = c_q 1 w_i 1 = 10^{\kappa_1} 1^2 \dots 1^2 0^{\kappa_m} 1^{2i+4},$$

$\overline{c_R}$ contains only one block of 1s of odd length, and that block has length $2i+5$.

Thus P_q^i is a period $n+2i+4$ orbit with rotation interval $[q, 1/2]$ and $r^{i-1}(P_q^i) = 1/2$. It will be shown that any horseshoe orbit with these properties must be P_q^i, which will complete the proof.

Suppose then that R is a period $n+2i+4$ periodic orbit with height $q(R) = q$, with $r^{i-1}(R) = 1/2$, and for which (using Theorem 56) $\overline{c_R}$ contains either the word 01010 or the word $01^{2m+1}0$ for some $m \geq 1$. Take

$$c_R = c_q 1 w' 1 = 10^{\kappa_1} 1^2 \dots 1^2 0^{\kappa_m} 1 w' 1$$

for some decoration w' of length $2i+1$ distinct from 1^{2i+1}. A contradiction will be derived.

Consider the first occurrence of one of the words 01010 or $01^{2m+1}0$ $(m \geq 1)$ in c_R^∞. Since $w' \neq 1^{2i+1}$, this word is either 01010 or $01^{2m+1}0$ for $1 \leq m \leq i$. Thus there is a corresponding shift of $\overline{c_R}$ of the form $\dots 01{}^{0}_{1}1^{2m-1}0 \cdot f$ where $1 \leq m \leq i$; or, in other words, of the form $b{}^{0}_{1}\tilde{v} \cdot f$, where $v = 1^{2m}$ is a non-trivial even initial subword of $w^{(i-1)+} = 1^{2i}$.

Because we chose the *first* occurrence of one of the words 01010 or $01^{2m+1}0$ in c_R^∞, $b = 10\dots$ cannot be of the form $101^{2k-1}0\dots$ for any $k \geq 1$. Thus $q(b) < 1/2$ by Lemma 57, and hence $\nu^{i-1}(R) < 1/2$.

Now $\mu^{i-1}(R) = \nu^{\hat{w}_{i-1}}(\hat{R}) = \nu^{i-1}(\hat{R})$. Since $q(\hat{R}) = q(R)$ (Lemma 3.8 of [Hal94]) and $\overline{c_{\hat{R}}}$ contains one of the words 01010 or $01^{2m+1}0$, this gives $\mu^{i-1}(R) < 1/2$. Hence $r^{i-1}(R) < 1/2$, which is the required contradiction.

□

Thus r^i is a braid type invariant for all $i \geq 0$. These invariants will now be used to prove that P_q^i has pseudo-Anosov braid type for all $i \geq 0$ and $q \in (0, 1/2)$. The proof will make use of a more general result (Theorem 62 below) for showing that horseshoe periodic orbits have pseudo-Anosov braid type.

The following result (Lemma 3.4 of [Hal94]) will be required:

Lemma 60. *Let $q \in (0, 1/2) \cap \mathbb{Q}$. Let w_q be as in Section 6.2. Then, for any $c \in \{0,1\}^{\mathbb{N}}$,*

$$q(c) = q \iff (10w_q1)^\infty \preceq c \preceq 10w_q0(11w_q0)^\infty.$$

In particular, if $q(c) = q$ then $c = 10w_q110\dots$ or $c = 10w_q01\dots$.

The first step is to give a lower bound on the period of horseshoe orbits R for which $r^i(R) < r^{i-1}(R)$.

Lemma 61. *Let $q = m/n \in (0, 1/2)$, and let R be a horseshoe periodic orbit such that $r^i(R) = q < r^{i-1}(R)$. Then R has period at least $n + 2i + 4$.*

Proof. a) Suppose first that $\lambda^i(R) = q$. It follows from Definitions 14 that for some s

$$\sigma^s(\overline{c_R}) = b{}^{0}_{1}1^{2i+1}{}^{0}_{1} \cdot f,$$

where $q(b) \leq q$ and $q(f) \leq q$, with equality in one of the two cases. Suppose that $q(f) = q$: the case where $q(b) = q$ works identically. Hence (Lemma 60) either $f = 10w_q110\dots$ or $f = 10w_q01\dots$. In the former case, $\overline{c_R}$ contains both the word $01{}^{0}_{1}1^{2i+1}{}^{0}_{1}10$ (which has length $2i+7$ and contains only isolated 0s and blocks of 1s of odd length) and the word $0w_q110$ (which has length $n+1$ and contains only blocks of 1s of even length). Thus R has period at least $(2i+7) + (n+1) - 2 > n + 2i + 4$ as required. In the latter case, $\overline{c_R}$ again contains the word $01{}^{0}_{1}1^{2i+1}{}^{0}_{1}10$, and also contains the word $0w_q0$ (which has length $n-1$ and contains only blocks of 1s of even length). Thus R has period

at least $(2i+7)+(n-1)-2 = n+2i+4$ as required. (Note that if R has period $n+2i+4$, then $\sigma^s(\overline{c_R}) = 10w_q01{}^0_11^{2i+1}{}^0_1 = c_q{}^0_11^{2i+1}{}^0_1$, i.e. $R = P_q^i$.)

b) Suppose, then, that $\lambda^i(R) \neq q$. Since $r^i(R) = q$, it follows from Definitions 14 that $\lambda^i(R) > q$, and that $\mu^i(R) \leq q$ and $\nu^i(R) \leq q$, with equality in one of the two cases. Suppose that $\mu_i(R) = q$: the case where $\nu_i(R) = q$ works identically. Hence, by Definitions 14, there is some s such that

$$\sigma^s(\overline{c_R}) = b01^{2k-1}{}^0_1 \cdot f,$$

where $1 \leq k \leq i+1$ and $q(f) = q$ (here $01^{2k-1} = \check{v}$, where $v = 1^{2k}$ is a non-empty even final subword of ${}^+w^i = 1^{2i+2}$).

i) If $k = i+1$, then $\overline{c_R}$ contains the word $01^{2i+1}{}^0_110$ (of length $2i+5$, with only isolated 0s and blocks of 1s of odd length). It also contains either the word $0w_q110$ (length $n+1$) or $0w_q0$ (length $n-1$), which contain only blocks of 1s of even length. So if R has period less than $n+2i+4$, $\overline{c_R}$ must contain the words $01^{2i+1}{}^0_110$ and $0w_q0$, and these words must overlap at either one or both of their endpoints: that is, $\sigma^s(\overline{c_R})$ is either $\overline{10w_q01^{2i+1}{}^0_1}$, or $\overline{10w_q001^{2i+1}{}^0_1}$, or $\overline{100w_q01^{2i+1}{}^0_1}$. In the first case, $R = P_q^{i-1}$, and hence $r^{i-1}(R) = q$, contradicting the hypothesis that $q < r^{i-1}(R)$. In the second case, $f = 10w_q00\ldots$ and in the third case, $f = 100w_q0\ldots$, each contradicting $q(f) = q$ (since by Lemma 60, if $q(f) = q$ then the number of 1s in the first $n+1$ symbols of f is either $2m$ or $2m+1$).

ii) If $k \leq i$, then $\mu^{i-1}(R) \leq q$ (since 1^{2k} is a non-empty even final subword of ${}^+w^{i-1}$). Since $r^{i-1}(R) > q$, it follows that $\nu^{i-1}(R) > q$.

Let $\nu^i(R) = r \leq q$. Then $\overline{c_R}$ contains the word $w_r01{}^0_11^{2i+1}0$ (the block of 1s here can't be shorter, since $\nu^{i-1}(R) > r$). In particular, it contains the word $01{}^0_11^{2i+1}0$ (which has length $2i+5$ and contains only isolated 0s and blocks of 1s of odd length). Since $\mu^i(R) = q$, $\overline{c_R}$ also contains either the word $0w_q110$ (length $n+1$) or $0w_q0$ (length $n-1$). So if R has period less than $n+2i+4$, $\overline{c_R}$ must contain the words $01{}^0_11^{2i+1}0$ and $0w_q0$, and these words must overlap at either one or both of their endpoints: that is, $\sigma^s(\overline{c_R})$ is either $\overline{10w_q01{}^0_11^{2i}}$, or $\overline{10w_q001{}^0_11^{2i}}$, or $\overline{100w_q01{}^0_11^{2i}}$. As before, in the first case $R = P_q^{i-1}$, contradicting $r^{i-1}(R) < q$, while the other two cases contradict $q(f) = q$.

□

Note that there is no restriction on the decoration of the horseshoe periodic orbit R in the following result, which thus provides a general test for pseudo-Anosov braid type of horseshoe periodic orbits.

Theorem 62. *Let $i \geq 1$ and let R be a period N horseshoe orbit with $r^i(R) = m/n < r^{i-1}(R)$. Let d be the largest divisor of N other than N itself. If $d < n+2i+4$, then R has pseudo-Anosov braid type.*

Proof. The fact that $r^i(R) < r^{i-1}(R) \leq 1/2$ means that R forces infinitely many periodic orbits of decoration w^i, so cannot be of finite order braid type.

If R had reducible braid type, then it would force the braid type of the outermost component in its Nielsen-Thurston canonical representative g: in particular, this is the braid type of some horseshoe periodic orbit S. Since $R \geq S$, it follows that $r^{i-1}(S) \geq r^{i-1}(R)$. Now $R \geq P_q^i$ for all $q > r^i(R)$ with $q \in \mathbb{Q}^{w_i}$, so g has periodic orbits of each of these braid types in its outermost component, and hence $S \geq P_q^i$ for all $q > r^i(R)$ with $q \in \mathbb{Q}^{w_i}$. So $r^i(S) = r^i(R) < r^{i-1}(R) \leq r^{i-1}(S)$, and hence S has period at least $n + 2i + 4$ by Lemma 61. This contradicts the fact that the period of S is at most d, which is less than $n + 2i + 4$. □

Corollary 63. *P_q^i has pseudo-Anosov braid type for all $i \geq 0$ and all $q \in (0, 1/2)$.*

Proof. Let $q = m/n$. Suppose first that $i > 0$. Then $r^{i-1}(P_q^i) = 1/2$, as established in the proof of Theorem 59; and $r^i(P_q^i) = q$ by Lemma 53 a). Since P_q^i has period $n + 2i + 4$, the result follows from Theorem 62.

For the case $i = 0$, suppose for a contradiction that R has reducible braid type. As in the proof of Theorem 62, let S be a horseshoe periodic orbit whose braid type is that of the outermost component in the Nielsen-Thurston canonical representative of the braid type of R. Then $\rho i(S) = \rho i(R)$, so in particular $q(S) = q(R) = m/n$, and hence S has period at least n. Since R has period $n + 4$, the period of S is at most $n/2 + 2$: thus $n \leq 4$. A direct check verifies that the orbits $P_{1/3}^0$ and $P_{1/4}^0$ have pseudo-Anosov braid type. □

It follows from Theorem 19 that $P_q^i \geq P_{q'}^i$ for all i and all $q, q' \in (0, 1/2)$ with $q \leq q'$. The forcing between families with different i can also be determined easily using the invariants r^i. The next result says that P_q^i forces none of the $P_{q'}^j$ with $j < i$, while if $j \geq i$ it forces all those with $q' > q$: in the language of Conjecture 8, this means that $w^j \preceq w^i$ if and only if $j \geq i$.

Theorem 64. *Let i and j be non-negative integers, and $q \in (0, 1/2) \cap \mathbb{Q}$. Then*

$$r^j(P_q^i) = \begin{cases} q & \textit{if } j \geq i \\ \frac{1}{2} & \textit{if } j < i. \end{cases}$$

Proof. Let $R = P_q^i$ with $c_R = c_q 1^{2i+3}$. Then the only words of the form $01^{2k+1}0$ in $\overline{c_R}$ have $k = i + 2$. It is therefore immediate from Lemma 58 that $r^j(R) = 1/2$ for $j < i$.

$r^i(R) = q$ by Lemma 53 a), so suppose that $j > i$. $r^j(R) \geq q$ by Lemma 53 b), so it is only necessary to show that $r^j(R) \leq q$. Now $\overline{c_R} = \dots 01^{2i+3}1 \cdot \left(c_q 1^{2i+3}\right)^\infty$, and $01^{2i+3}1$ is a word of the form $\check{v}_1^0$, where $v = 1^{2i+4}$ is a non-empty even final subword of ${}^+w^j = 1^{2j+2}$. Hence $\mu^j(R) \leq q(\left(c_q 1^{2i+3}\right)^\infty) = q$. Similarly $\nu^j(R) \leq q$, giving $r^j(R) \leq q$ as required. □

Corollary 65. *Let R be a period N horseshoe orbit. Then $(r^i(R))$ is a decreasing sequence, with $r^i(R) = r^{i'}(R)$ if $i, i' \geq \lfloor (N-7)/2 \rfloor$.*

Proof. Let $q > r^i(R)$ and pick $q' \in (r^i(R), q)$. Then $R \geq P^i_{q'}$ by Theorem 19, and $P^i_{q'} \geq P^{i+1}_q$ by Theorem 64. That is, $R \geq P^{i+1}_q$ for all $q > r^i(R)$, and so $r^{i+1}(R) \leq r^i(R)$ as required.

That the sequence $(r^i(R))$ stabilises after $i = \lfloor (N-7)/2 \rfloor$ is immediate from Lemma 61. $\square$

6.4. **Topological entropy bounds.** Recall that the *topological entropy* $h(\beta)$ of a braid type β is the minimum topological entropy of orientation-preserving homeomorphisms of the disk having a periodic orbit of braid type β: it is realised by the Nielsen-Thurston canonical representative of the braid type.

Let w be a lone decoration, and let $h^w(q) = h(P^w_q)$ denote the topological entropy of the braid type of the periodic orbits with height q and decoration w $(0 < q < q_w)$. It is clear that, for any horseshoe periodic orbit R, $h(R) \geq h^w(q)$ for all $q > r^w(R)$ and, in particular, that $h(R) \geq \bar{h}^w(r^w(R))$, where

$$\bar{h}^w(q) = \lim_{q' \searrow q} h^w(q').$$

It is often possible to calculate $\bar{h}^w(q)$ explicitly using train track techniques, providing a convenient means to compute topological entropy bounds. The approach for the decorations w_i of Section 6.3 will be outlined in this section. A similar calculation could in principle be carried out for the star decorations of Section 6.2, using the explicit train track maps described in [dCH04].

An explicit train track and train track map for the periodic orbits $P^{w_i}_q$ is depicted in Figure 13. Writing $q = m/n$, the $n + 2i + 4$ points of the orbit are depicted with solid circles. There are two valence $i + 3$ vertices, depicted with unfilled circles. The strings of edges denoted A, B, and C contain respectively $n - 2m + 1$, m, and m points of the orbit (the remaining $2i+3$ points comprising i at valence 1 vertices around the left hand valence $i + 3$ vertex, $i + 1$ at valence 1 vertices around the right hand valence $i + 3$ vertex, and 2 between these two vertices).

A routine but long calculation using this train track map shows that $h^{w_i}(m/n)$ is the logarithm of the largest real root of the polynomial

$$H^i_{m/n}(x) = x^{n+1} g_i(x) + 2x(x^2 - 1)(x^{2i+4} + 1) f_{m/n}(x) - x^{2i+6} g_i(1/x),$$

where

$$g_i(x) = x^{2i+3}(x^3 - x^2 - x - 1) - 2 \quad \text{and } f_{m/n}(x) = \sum_{j=1}^{m-1} x^{\lfloor jn/m \rfloor}.$$

(In fact $H^i_{m/n}(x)$ is $(x^2 - 1)$ times the characteristic polyomial of the transition matrix of the train track map.)

Using this result, the following theorem can be proved:

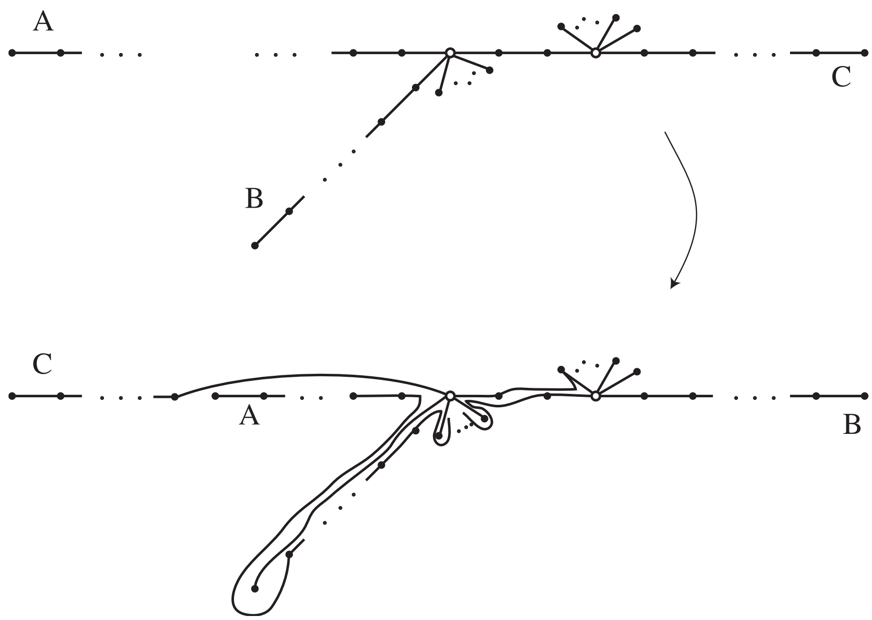

FIGURE 13. The train track map for $P_q^{w_i}$

Theorem 66. *Let $0 < m/n < 1/2$. The polynomial*

$$\overline{H}^i_{m/n}(x) = (x^n - 1)g_i(x) + 2(x^2 - 1)(x^{2i+4} + 1)(1 + f_{m/n}(x))$$

has a single real root $\lambda^i_{m/n}$ in $x > 1$, and $\bar{h}^{w_i}(m/n) = \log \lambda^i_{m/n}$.

Proof. Let

$$K(x) = \frac{\overline{H}^i_{m/n}(x)}{x-1} = g_i(x) \sum_{j=0}^{n-1} x^j + 2(x+1)(x^{2i+4} + 1)(1 + f_{m/n}(x)).$$

Observe that $K(1) = 4(2m - n) < 0$ and that $K(2) > 0$ since $g_i(2) \geq 0$, so that K has at least one root in $(1, 2)$. Now

$$\begin{aligned} K(x) &= x^{2i+6} \sum_{j=0}^{n-1} x^j + 2(x+1)(x^{2i+4} + 1) \sum_{j=0}^{n-1} x^{\lfloor jn/m \rfloor} - (x^{2i+5} + x^{2i+4} + x^{2i+3} + 2) \sum_{j=0}^{n-1} x^j \\ &= x^{n+2i+5} + 2(x+1)(x^{2i+4} + 1) \sum_{j=0}^{m-1} x^{\lfloor jn/m \rfloor} - x^{2i+5} - (x^{2i+4} + x^{2i+3} + 2) \sum_{j=0}^{n-1} x^j. \end{aligned}$$

Since $n > 2m$, every term of $2(x+1)(x^{2i+4}+1) \sum_{j=0}^{m-1} x^{\lfloor jn/m \rfloor}$ is cancelled by terms of $(x^{2i+4} + x^{2i+3} + 2) \sum_{j=0}^{n-1} x^j$, so that $K(x)$ is of the form

$$K(x) = x^{n+2i+5} - \sum_{j=0}^{n+2i+3} a_j x^j,$$

where all of the coefficients a_j are non-negative. It follows that all of its positive roots occur with positive derivative, so that $K(x)$ has a unique positive root, and hence $\overline{H}^i_{m/n}$ has a unique root in $(1,\infty)$ as required.

Now $\bar{h}^{w_i}(m/n)$ is the limit as $k\to\infty$ of the increasing sequence $\mu_k = h^{w_i}\left(\frac{m^2k}{mnk-1}\right)$. Pick a rational $m'/n' \in (0,1/2)$ which is greater than m/n, and let $\mu = h^{w_i}(m'/n')$, so that $\mu < \mu_k$ for all sufficiently large k. Restrict to such large k, and work with values of x in the interval $[e^\mu, 2]$. Then

$$\begin{aligned} H^i_{m^2k/(mnk-1)}(x) &= x^{mnk} g_i(x) + 2x(x^2-1)(x^{2i+4}+1)\sum_{j=1}^{m^2k-1} x^{\left\lfloor \frac{(m^2k-j)(mnk-1)}{m^2k}\right\rfloor} - x^{2i+6} g_i(1/x) \\ &= x^{mnk}\left(g_i(x) + 2(x^2-1)(x^{2i+4}+1)\sum_{j=1}^{m^2k-1} x^{\left\lfloor \frac{-j(mnk-1)}{m^2k}\right\rfloor} - x^{2i+6-mnk} g_i(1/x)\right). \end{aligned}$$

Now

$$\sum_{j=1}^{m^2k-1} x^{\left\lfloor \frac{-j(mnk-1)}{m^2k}\right\rfloor} = \sum_{r=0}^{mk-1} x^{-rn} \sum_{j=1}^{m-1} x^{\left\lfloor \frac{-jn}{m} + \frac{rm+j}{m^2k}\right\rfloor} + \sum_{r=1}^{mk-1} x^{-rn},$$

and since $(rm+j)/m^2k < 1/m$ for $r \le k-1$ and $j \le m-1$; and $(rm+j)/m^2k < 1$ for $r \le mk-1$ and $j \le m-1$, this gives

$$\begin{aligned} \sum_{j=1}^{m^2k-1} x^{\left\lfloor \frac{-j(mnk-1)}{m^2k}\right\rfloor} &= \sum_{r=0}^{k-1} x^{-rn} \sum_{j=1}^{m-1} x^{\lfloor -jn/m\rfloor} + \sum_{r=1}^{mk-1} x^{-rn} + R_{k,m/n}(x) \\ &= f_{m/n}(x)\sum_{r=1}^{k} x^{-rn} + \sum_{r=1}^{mk-1} x^{-rn} + R_{k,m/n}(x), \end{aligned}$$

where the remainder term $R_{k,m/n}(x)$ satisfies

$$0 \le R_{k,m/n}(x) \le x f_{m/n}(x) \sum_{r=k+1}^{mk} x^{-rn} \qquad \text{for all } x \in [e^\mu, 2].$$

Thus

$$\frac{(x^n-1)H^i_{m^2k/(mnk-1)}(x)}{x^{mnk}} = \overline{H}^i_{m/n}(x) + S_{k,i,m/n}(x),$$

where $S_{k,i,m/n}(x) \to 0$ as $k\to\infty$ uniformly for $x\in[e^\mu,2]$. For each k this function has a zero at e^{μ_k}, and hence the unique zero in $(1,\infty)$ of $\overline{H}^i_{m/n}(x)$ is at $\lim_{k\to\infty} e^{\mu_k} = e^{\bar{h}^{w_i}(m/n)}$ as required. □

Example 67. Note that entropy bounds obtained in this way depend only on local features of the code of the periodic orbit under consideration.

For example, let R be any horseshoe periodic orbit for which $\overline{c_R}$ contains the word $001{}^0_1111{}^0_1100$. Then some shift of $\overline{c_R}$ is of the form $bv\cdot f$, where $v = {}^0_1111{}^0_1 = {}^0_1w_1{}^0_1$,

and $q(b) \le 1/3$, $q(f) \le 1/3$. Thus $r^{w_1}(R) \le 1/3$, and hence $h(R) \ge \bar{h}^{w_1}(1/3)$, which is the logarithm of the unique root in $x > 1$ of the polynomial $\overline{H}^1_{1/3}(x)$.

Now $\overline{H}^1_{1/3}(x) = (x^3-1)g_1(x) + 2(x^2-1)(x^6+1)$ (note that $f_{1/3}(x) = 0$), which simplifies (using $g_1(x) = x^5(x^3-x^2-x-1)-2$) to

$$\overline{H}^1_{1/3}(x) = x^2\left(x^4-1\right)\left(x^5-x^4-x^3+2\,x-2\right).$$

Thus any periodic orbit R whose code contains this word has $h(R) \ge \log(1.47669)$.

Compare this to the topological entropy of $P^{w_1}_{1/3}$ itself, which is given by the largest positive root of the polynomial

$$\begin{aligned} H^1_{1/3}(x) &= x^4 g_1(x) - x^8 g_1(1/x) \\ &= \left(x^4-1\right)\left(x^8-x^7-x^6-x^5+3\,x^4-x^3-x^2-x+1\right), \end{aligned}$$

giving $h^{w_1}_{1/3} \simeq \log(1.56294)$.

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